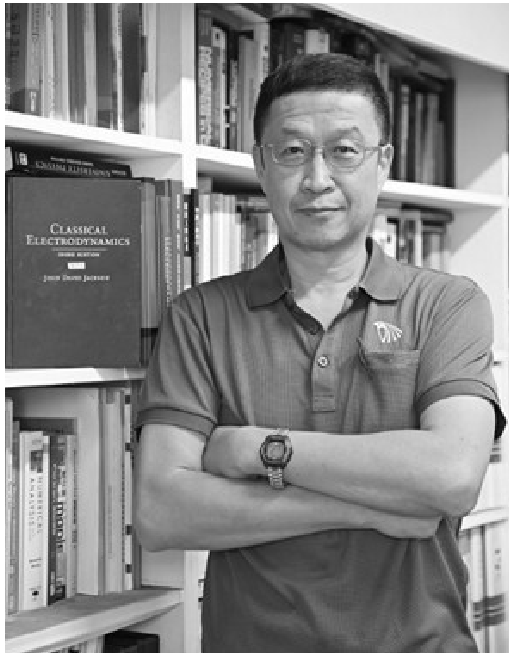
Chapter 3: Boundary-Value Problems in Electrostatics: II

Dirac said: “I understand what an equation means if I have a way of figuring out the characteristics of its solution without actually solving it.”

A physical understanding is a completely un-mathematical, imprecise, and inexact thing, but absolutely necessary for a physicist.

The Feynman Lectures on Physics II, Chap.2

Dirac and Feynman

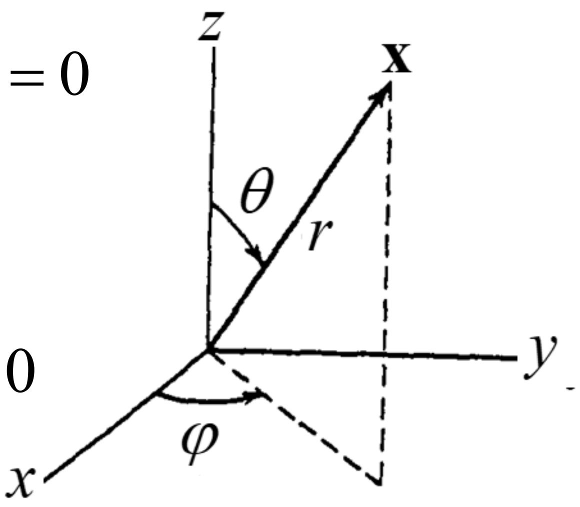


3.1 Laplace Equation in Spherical Coordinates

$$\nabla^2\Phi(\mathbf{x})=0$$

or $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right)$

$$\Rightarrow \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2}=0$$



Let $\Phi(\mathrm{x})=\frac{U(r)}{r}P(\theta)Q(\varphi)$

$$\Rightarrow PQ\frac{d^2U}{dr^2}+\frac{UQ}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)+\frac{UP}{r^2\sin^2\theta}\frac{d^2Q}{d\varphi^2}=0$$

Multiply by $\frac{r^2\sin^2\theta}{UPQ}$

Dividing all terms by $\sin^2\theta$, we see that the r-dependence is isolated within this term. So this term must be a constant. Let it be $\nu(\nu+1)$.

$$\Rightarrow \sin^2\theta\,[\overbrace{\frac{1}{U}r^2\frac{d^2U}{dr^2}}^{=\nu(\nu+1)}+\frac{1}{P\sin\theta}\frac{d}{d\theta}(\sin\theta\frac{dP}{d\theta})]+\underbrace{\frac{1}{Q}\frac{d^2Q}{d\varphi^2}}_{=-m^2}=0 \qquad (3.3)$$

The φ-dependence is isolated within this term, so this term must be a constant. Let it be $-m^2$.

3.1 Laplace Equation in Spherical Coordinates (*continued*)

Rewrite (3.3): $$\sin^2\theta\,[\underbrace{\frac{1}{U}r^2\frac{d^2U}{dr^2}}_{=\nu(\nu+1)}+\frac{1}{P\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)]+\overbrace{\frac{1}{Q}\frac{d^2Q}{d\varphi^2}}^{=-m^2}=0$$

The equation for $Q(\varphi)$ is: $$\frac{d^2Q}{d\varphi^2}+m^2Q=0 \qquad (3.4)$$

$$\Rightarrow Q=e^{im\varphi},e^{-im\varphi}$$

m is to be determined from the b.c.

The equation for $P(\theta)$ is

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)+\left[\nu(\nu+1)-\frac{m^2}{\sin^2\theta}\right]P=0. \qquad (3.6)$$

Let $x=\cos\theta$, then the equation takes the form of the associated Legendre equation:

$$\frac{d}{dx}\left(1-x^2\right)\frac{dP}{dx}+\left[\nu\left(\nu+1\right)-\frac{m^2}{1-x^2}\right]P=0$$

$$\Rightarrow P=\begin{Bmatrix}P_\nu^m(x)\\ Q_\nu^m(x)\end{Bmatrix}=\begin{Bmatrix}P_\nu^m(\cos\theta)\\ Q_\nu^m(\cos\theta)\end{Bmatrix} \qquad (2)$$

ν is to be determined from the b.c.

Rewrite (3.3): $$\sin^2\theta\,[\overbrace{\frac{1}{U}r^2\frac{d^2U}{dr^2}}^{=\nu(\nu+1)}+\frac{1}{P\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)]+\overbrace{\frac{1}{Q}\frac{d^2Q}{d\varphi^2}}^{=-m^2}=0$$

The equation for $U(r)$ is: $$\frac{d^2U}{dr^2}-\frac{\nu(\nu+1)}{r^2}U=0 \qquad (3.7)$$

Since ν is determined from the b.c. for (3.6), this is not an eigenvalue problem.

$$\Rightarrow U=r^{\nu+1},r^{-\nu}\Rightarrow\frac{U}{r}=r^{\nu},r^{-\nu-1}$$

Thus,

$$\Phi=\begin{Bmatrix}r^{\nu}\\ r^{-\nu-1}\end{Bmatrix}\begin{Bmatrix}P_\nu^m(\cos\theta)\\ Q_\nu^m(\cos\theta)\end{Bmatrix}\begin{Bmatrix}e^{im\varphi}\\ e^{-im\varphi}\end{Bmatrix},$$

where each bracket represents a linear combination of the two functions inside. Because the differential equation is linear, the linear combination of any number of solutions is also a solution.

Note that ν and m are arbitrary constants until we apply boundary conditions.

3.2 Legendre Equation and Legendre Polynomials

[lə'ʒaŋdɚ]

We will then deal with the mathematical properties (Secs. 3.2, 3.5, & 3.6).

Legendre Equation :

$$\frac{d}{dx}\left[\left(1-x^2\right)\frac{du}{dx}\right]+\nu\left(\nu+1\right)u=0, \quad -1\le x\le 1 \tag{3.9}$$

The solutions are: $u(x)=AP_\nu(x)+BQ_\nu(x)$

$$\begin{cases} P_\nu(x): \ \underline{\text{Legendre function of the first kind}} \\ Q_\nu(x): \ \underline{\text{Legendre function of the second kind}} \end{cases}$$

Ref. 1: Gradshteyn & Ryzhik, "Table of Integrals, Series, and Products," Chs. 7 & 8.

Ref. 2: Abramowitz & Stegun, "Handbook of Mathematical Functions," Ch. 8.

Rewrite the solution: $u(x) = AP_\nu(x) + BQ_\nu(x)$

$Q_\nu(x)$ diverges as $x \to \pm 1$. Hence, $Q_\nu(x)$ is not commonly used in physics.

$P_\nu(x)$ is finite for $|x| < 1$ and $x = 1$, but $P_\nu(-1)$ diverges unless ν is an integer (see p.105.)

In many physics problems, boundary conditions require ν to be an integer. Since the form of the Legendre equation is unchanged if $\nu \to -\nu - 1$, we have $P_{-\nu-1}(x) = P_\nu(x)$. Hence, when ν is an integer (denoted by l), negative l is redundant. Thus, $l = 0,\ 1,\ 2\cdots$ and $P_l(x)$ becomes a polynomial (properties on following pages).

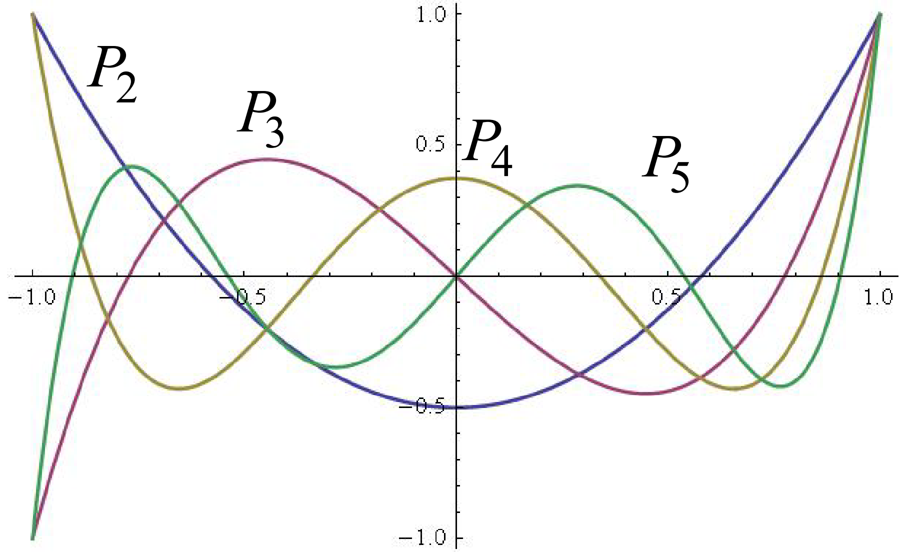
Note: The range $(-1 \le x \le 1)$ considered here is often encountered in physics problems. Mathmatically, the range of $P_\nu(x)$ and $Q_\nu(x)$ can be extended to the entire complex $x + iy$ plane. Furthermore, ν can also be a complex number (See Gradshteyn & Ryzhik).

3.2 Legendre Equation and Legendre Polynomials (*continued*)

Legendre Polynomial : $P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l,\ \ l = 0,\ 1,\ 2\ldots\ (3.16)$

$$\boxed{P_l(-x) = (-1)^l P_l(x)}$$

$$\boxed{P_l(-1) = (-1)^l}\qquad \boxed{P_l(1) = 1}\qquad \boxed{Q_l(1) \to \infty}$$

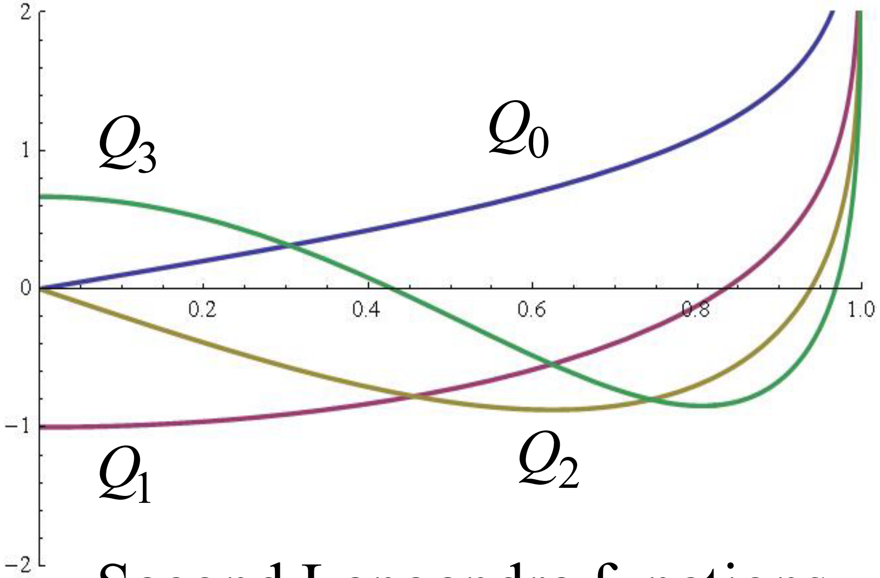


Lengendre polynomials $P_2(x)$ to $P_5(x)$

$P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = (3x^2-1)/2$,

$P_3(x) = (5x^3-3x)/2$,

$P_4(x) = (35x^4-30x^2+3)/2$



Second Lengendre functions $Q_0(x)$, $Q_1(x)$, $Q_2(x)$, and $Q_3(x)$

The set $P_l(x)$ is orthogonal: $\int_{-1}^{1} P_{l'}(x)P_l(x)dx = \frac{2}{2l+1}\delta_{l'l}$ (3.21)

It is also complete in index l. Hence, any function $f(x)$ can be expanded as

$$f(x) = \sum_{l=0}^{\infty} A_l P_l(x) \qquad (3.23)$$

3.5 Associated Legendre Functions and the Spherical Harmonics

Associated Legendre Equation :

$$\frac{d}{dx}\left(1-x^2\right)\frac{du}{dx} + \left[\nu\left(\nu+1\right) - \frac{m^2}{1-x^2}\right]u = 0 \text{ , for } -1 \le x \le 1$$

The solutions are: $u(x) = AP_\nu^m(x) + BQ_\nu^m(x)$

$$\begin{cases} P_\nu^m : \ \underline{\text{associated Legendre function of the first kind}} \\ Q_\nu^m : \ \underline{\text{associated Legendre function of the second kind}} \end{cases}$$

(Refs.: Gradshteyn & Ryzhik; Abramowitz & Stegun; Eisberg & Resnick, Quantum Physics)

Rewrite the solution: $u(x) = AP_\nu^m(x) + BQ_\nu^m(x)$

$Q_\nu^m(x)$ diverges as $x \to \pm 1$, hence is not commonly used in physics.

$P_\nu^m(x)$ is finite on the interval $-1 \le x \le 1$ only when

$$\begin{cases} \nu \text{ is zero or a positive integer } (\nu = l = 0,\ 1,\ 2\ldots) \text{ and} \\ m = -l,\ -(l-1), \ldots,\ -1,\ 0,\ 1, \ldots,\ (l-1),\ l \end{cases}$$ [p. 107.]

Under these conditions, we have (for positive or negative m)

$$P_l^m(x) = \frac{(-1)^m}{2^l\, l!}(1-x^2)^{\frac{m}{2}}\left(\frac{d}{dx}\right)^{l+m}(x^2-1)^l = (1-x^2)^{\frac{m}{2}}\frac{d^m P_l(x)}{dx^m} \qquad (3.50)$$

with the properties:
$$\begin{cases} P_l^0(x) = P_l(x) \\ P_l^m(-x) = (-1)^{l+m} P_l^m(x) \\ P_l^{-m}(x) = (-1)^m \dfrac{(l-m)!}{(l+m)!} P_l^m(x) & (3.51) \\ \int_{-1}^{1} P_{l'}^m(x) P_l^m(x)\,dx = \dfrac{2}{2l+1}\dfrac{(l+m)!}{(l-m)!}\delta_{ll'} & (3.52) \end{cases}$$

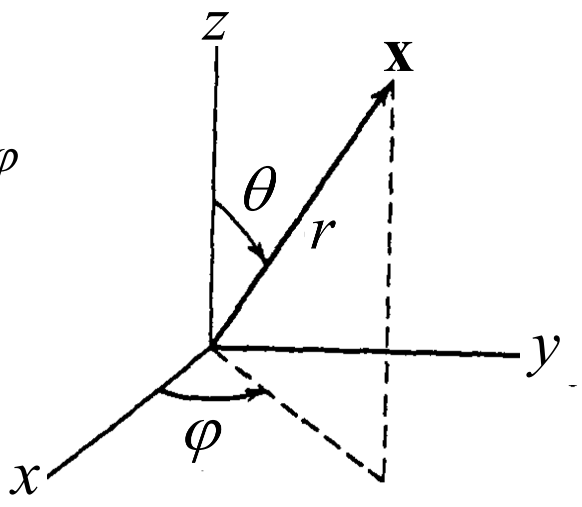
The set $P_l^m(x)$ is complete in index l in the sense any function $f(x)$ can be expanded as $f(x)=\sum_{l=|m|}^{\infty} C_l P_l^m(x)$ $\left[\begin{array}{l} m: \text{ a fixed integer} \\ \text{See (A.3) in Appendix A.} \end{array}\right]$

Spherical Harmonics $Y_{lm}(\theta,\varphi)$:

$$Y_{lm}(\theta,\varphi)\equiv\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{im\varphi}, \qquad (3.53)$$

where $l=0$ or a positive integer; $m=-l,\ -(l-1),\ldots,\ 0,\ldots,\ (l-1),\ l$

$$\text{Examples}:\begin{cases} Y_{0,0}(\theta,\varphi)=\sqrt{\frac{1}{4\pi}} \\ Y_{1,-1}(\theta,\varphi)=\sqrt{\frac{3}{8\pi}}\sin\theta e^{-i\varphi} \\ Y_{1,0}(\theta,\varphi)=\sqrt{\frac{3}{4\pi}}\cos\theta \\ Y_{1,1}(\theta,\varphi)=-\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\varphi} \end{cases}$$



Properties of spherical harmonics:

(i) Using the orthogonality relation,

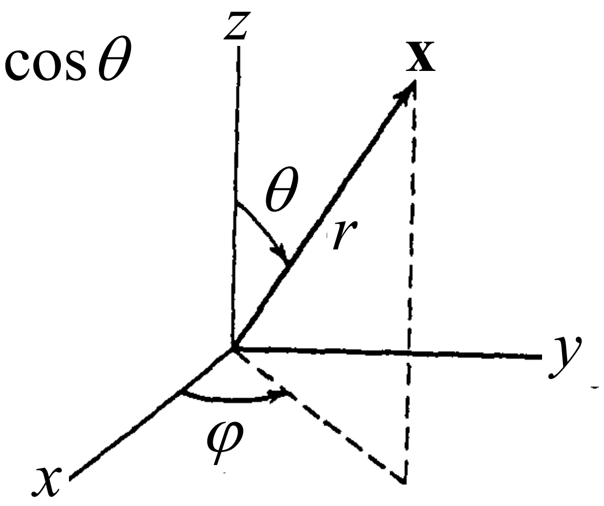
$$\int_{-1}^{1} P_{l'}^{m}(x)P_{l}^{m}(x)dx = \frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}\delta_{ll'} \qquad (3.52)$$

we can show that the spherical harmonics are orthonormal, i.e.,

$$\int d\Omega Y_{l'm'}^{*}(\theta,\varphi)Y_{lm}(\theta,\varphi) = \delta_{ll'}\delta_{mm'}, \qquad (3.55)$$

where

$$\int d\Omega = \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta d\theta = \int_0^{2\pi} d\varphi \int_{-1}^{1} d\cos\theta$$



(ii) The set $Y_{lm}(\theta,\varphi)$ is complete, i.e., an arbitrary function $g(\theta,\varphi)$ can be expanded as

$$g(\theta,\varphi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}A_{lm}Y_{lm}(\theta,\varphi) \qquad (3.58)$$

Multiplying both sides by $Y_{lm}^{*}(\theta,\varphi)$, integrating over θ,φ, and making use of (3.55), we obtain

$$A_{lm}=\int d\Omega Y_{lm}^{*}(\theta,\varphi)g(\theta,\varphi)$$

Substitution of A_{lm} into (3.58) gives the following expression for $g(\theta,\varphi)$,

$$g(\theta,\varphi)=\int d\Omega'\left[\sum_{l=0}^{\infty}\sum_{m=-l}^{l}Y_{lm}^{*}(\theta',\varphi')Y_{lm}(\theta,\varphi)\right]g(\theta',\varphi')$$

$$\Rightarrow \sum_{l=0}^{\infty}\sum_{m=-l}^{l}Y_{lm}^{*}(\theta',\varphi')Y_{lm}(\theta,\varphi)=\delta(\varphi-\varphi')\delta(\cos\theta-\cos\theta') \qquad (3.56)$$

This is the completeness relation of $Y_{lm}(\theta,\varphi)$ [cf. (2.34) & (2.35).]

(iii) Other properties of $Y_{lm}(\theta,\varphi)$:

$$\begin{cases} Y_{l,-m}(\theta,\varphi) = (-1)^m Y_{lm}^*(\theta,\varphi) \\ Y_{l,0}(\theta,\varphi) = \sqrt{\dfrac{2l+1}{4\pi}} P_l(\cos\theta) \end{cases}$$

This can be seen from the definition of $Y_{lm}(\theta,\varphi)$:

$$Y_{lm}(\theta,\varphi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta)e^{im\varphi}$$

and the relations:

$$P_l^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(x) \tag{3.51}$$

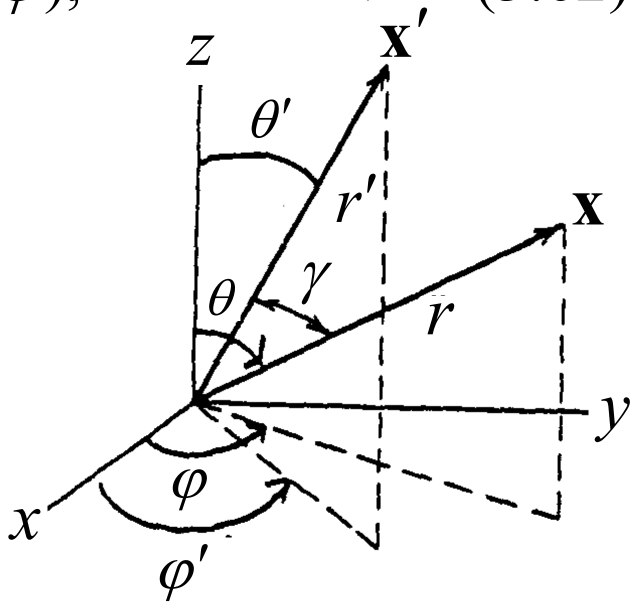
$$P_l^0(x) = P_l(x)$$

3.6 Addition Theorem for Spherical Harmonics

The addition theorem for spherical harmonics is derived on pp. 110-111. Here we write the theorem without derivation:

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi), \qquad (3.62)$$

where γ is the angle between $\mathbf{x}$ and $\mathbf{x}'$.



Setting $l = 1$ in (3.62) gives

$$P_1(\cos\gamma) = \frac{4\pi}{3}[Y_{1,-1}^*(\theta',\varphi')Y_{1,-1}(\theta,\varphi) + Y_{1,0}^*(\theta',\varphi')Y_{1,0}(\theta,\varphi) + Y_{1,1}^*(\theta',\varphi')Y_{1,1}(\theta,\varphi)]$$

Using $P_1(\cos\gamma) = \cos\gamma$, $Y_{1,-1} = \sqrt{\frac{3}{8\pi}}\sin\theta e^{-i\varphi}$, $Y_{1,0} = \sqrt{\frac{3}{4\pi}}\cos\theta$,

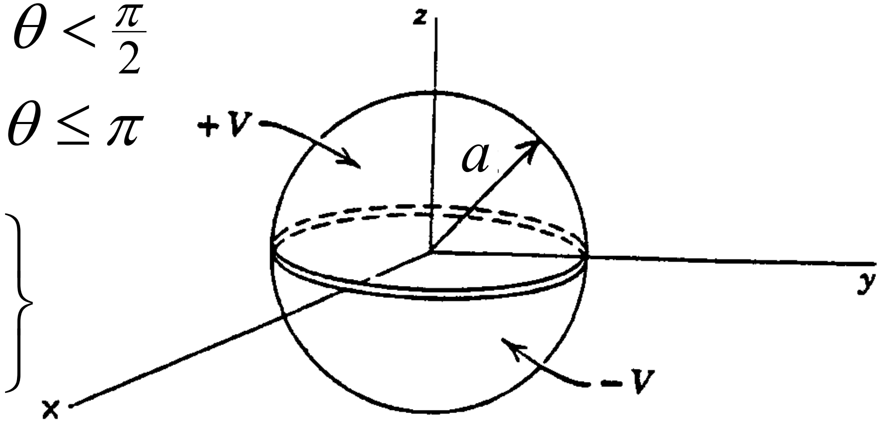
and $Y_{1,1} = -\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\varphi}$, we obtain a useful expression:

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi - \varphi'). \qquad (1)$$

3.3 Boundary-Value Problems with Azimuthal Symmetry

Example 1: Find Φ inside 2 hemispheres held at opposite potentials as shown in the figure.

$$\nabla^2\Phi = 0, \quad \Phi(a,\theta) = \begin{cases} V, & 0 \le \theta < \frac{\pi}{2} \\ -V, & \frac{\pi}{2} \le \theta \le \pi \end{cases}$$



Sol :

$$\Phi = \begin{Bmatrix} r^{\nu} \\ r^{-\nu-1} \end{Bmatrix} \begin{Bmatrix} P_{\nu}^{m}(\cos\theta) \\ Q_{\nu}^{m}(\cos\theta) \end{Bmatrix} \begin{Bmatrix} e^{im\varphi} \\ e^{-im\varphi} \end{Bmatrix}$$

(i) Φ is independent of φ. $\Rightarrow m = 0$

(ii) Φ is finite at $\theta = 0$ and π (i.e., at $\cos\theta = 1$ and -1).

$\Rightarrow \ \nu = l = 0, 1, 2, \ldots$ and drop Q_{ν}^{m}

(iii) Φ is finite at $r = 0$. $\Rightarrow$ drop $r^{-\nu-1}$

$$\Rightarrow \Phi(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

Note :

1. $P_{\nu}(-1)$ diverges unless ν is an integer (p.105.)
2. We have set $l = 0, 1, 2, \cdots$ because $P_{-l-1}(x) = P_l(x)$.
3. $Q_{\nu}(x) \to \infty$ as $x \to \pm 1$.

3.3 Boundary-Value Problems with Azimuthal Symmetry (*continued*)

The b.c. at $r=a$ is: $\Phi(a,\theta)=\sum_{l} A_l a^l P_l(\cos\theta)=\begin{cases} V, & 0\le\theta<\frac{\pi}{2} \\ -V, & \frac{\pi}{2}\le\theta\le\pi \end{cases}$

Use $\int_{-1}^{1} P_l(x)P_{l'}(x)dx=\frac{2}{2l+1}\delta_{ll'}$ (3.21)

$$\Rightarrow \int_{\pi}^{0} P_l(\cos\theta)\Phi(a,\theta)d\cos\theta = A_l a^l \overbrace{\int_{\pi}^{0} P_l^2(\cos\theta)d\cos\theta} = A_l a^l \frac{2}{2l+1}$$

$$\Rightarrow A_l=\frac{V}{a^l}\frac{2l+1}{2}\left[\int_{\frac{\pi}{2}}^{0} P_l(\cos\theta)d\cos\theta-\int_{\pi}^{\frac{\pi}{2}} P_l(\cos\theta)d\cos\theta\right]$$

$$=\begin{cases} \frac{V}{a^l}\frac{2l+1}{2}\frac{\left(-\frac{1}{2}\right)^{\frac{l-1}{2}}(l-2)!!}{\left(\frac{l+1}{2}\right)!}, & \text{for odd } l \\ 0, & \text{for even } l \end{cases}$$

$(2n+1)!!=(2n+1)(2n-1)(2n-3)\ldots5\times3\times1$

!! double factorial

pp. 99-100

$$\Rightarrow \Phi(r,\theta)=V\left[\frac{3}{2}\frac{r}{a}P_1(\cos\theta)-\frac{7}{8}\left(\frac{r}{a}\right)^3P_3(\cos\theta)+\cdots\right],\quad r\le a \qquad (3.36)$$

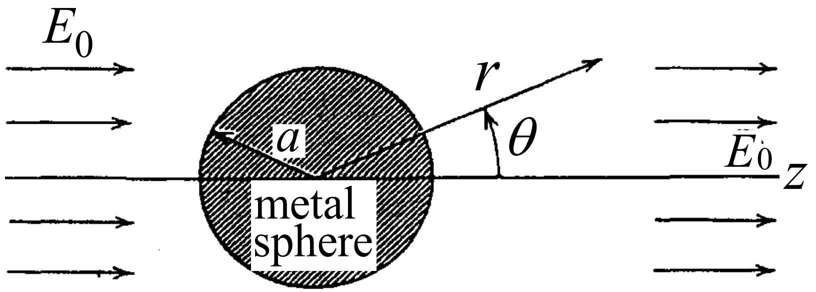
To find Φ for $r>a$, replace $\left(\frac{r}{a}\right)^l$ in (3.36) by $\left(\frac{a}{r}\right)^{l+1}$ [see (2.27)]

3.3 Boundary-Value Problems with Azimuthal Symmetry (*continued*)

Example 2: A conducting sphere of radius a with net charge Q on its surface is placed in a uniform electric field $E_0\mathbf{e}_z$. Use the method of expansion to find Φ outside the sphere and σ on the sphere.

Sol:

$$\Phi = \begin{Bmatrix} r^{\nu} \\ r^{-\nu-1} \end{Bmatrix} \begin{Bmatrix} P_\nu^m(\cos\theta) \\ Q_\nu^m(\cos\theta) \end{Bmatrix} \begin{Bmatrix} e^{im\varphi} \\ e^{-im\varphi} \end{Bmatrix}$$



(i) Φ is independent of φ. $\Rightarrow m = 0$

(ii) Φ is finite at $\theta = 0$ and π (i.e., at $\cos\theta = 1$ and -1).

$\Rightarrow \ \nu = l = 0,\ 1,\ 2, \ldots$ and drop Q_ν^m

Hence, $\Phi(r,\theta) = \sum\limits_{l=0}^{\infty} [A_l r^l + B_l r^{-(l+1)}] P_l(\cos\theta)$

b.c. at $r \to \infty$: $\Phi = \underbrace{-E_0 r\cos\theta}_{\text{external field}} + \frac{1}{4\pi\varepsilon_0} \underbrace{\frac{Q}{r}}_{\text{due to net chrage } Q} + \cdots$

$\Rightarrow A_1 = -E_0,\ A_l = 0$ (for $l \neq 1$), and $B_0 = \frac{1}{4\pi\varepsilon_0} Q$

Question:

$r^l \to \infty$ ($l \geq 1$) as $r \to \infty$.

Why keep the $A_l r^l$ terms?

There is a **catch** to this question.

$$\Rightarrow \Phi(r,\theta) = -E_0 r\cos\theta + \frac{1}{4\pi\varepsilon_0}\frac{Q}{r} + \sum_{l=1}^{\infty} B_l r^{-(l+1)} P_l(\cos\theta)$$

b.c. at $r = a$: $\Phi(r = a) = const.$

$$\Rightarrow \Phi(r = a) = \underbrace{\left(-E_0 a + \frac{B_1}{a^2}\right)}_{0}\underbrace{\cos\theta}_{\substack{\text{not a}\\\text{const.}}} + \frac{1}{4\pi\varepsilon_0}\frac{Q}{a} + \sum_{l=2}^{\infty} \underbrace{B_l}_{0}\, a^{-(l+1)}\underbrace{P_l(\cos\theta)}_{\text{not a const.}}$$

$$\Rightarrow\ B_1 = E_0 a^3 \text{ and } B_l = 0 \text{ for } l \ge 2$$

$$\Rightarrow\ \Phi(r,\theta) = -E_0 r\cos\theta + \frac{1}{4\pi\varepsilon_0}\frac{Q}{r} + \underbrace{E_0\frac{a^3}{r^2}\cos\theta}_{\substack{\text{due to induced surface charge}\\ \text{density } \sigma \text{ on the sphere}}}$$

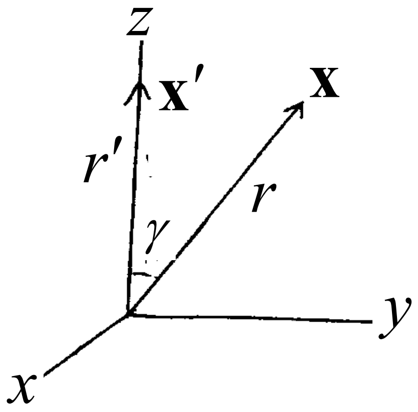
As will become clear in Ch. 4 [Eq. (4.56)], the $E_0\dfrac{a^3}{r^2}\cos\theta$ term in Φ is due to an electric dipole of dipole moment $p = 4\pi\varepsilon_0 a^3 E_0$. (see p.64)

The induced surface charge density σ is

$$\sigma = -\varepsilon_0 \frac{\partial\Phi}{\partial r}\bigg|_{r=a} = 3\varepsilon_0 E_0\cos\theta + \frac{Q}{4\pi a^2}$$

Example 3: Φ due to a unit point source at $\mathbf{x}'$ in infinite space

First, let's assume the point source is on the z-axis (at a distance r' from the origin) and divide the space into two regions: $r < r'$ and $r > r'$. In each region, we have $\nabla^2\Phi = 0$ with the solution



Sol:

$$\Phi = \begin{Bmatrix} r^{\nu} \\ r^{-\nu-1} \end{Bmatrix} \begin{Bmatrix} P_\nu^m(\cos\gamma) \\ Q_\nu^m(\cos\gamma) \end{Bmatrix} \begin{Bmatrix} e^{im\varphi} \\ e^{-im\varphi} \end{Bmatrix}$$

(i) Φ is indep. of φ. $\Rightarrow\ m = 0$

(ii) Φ is finite at $\gamma = 0$ and π. $\Rightarrow \nu = l = 0, 1, 2, \ldots$ and drop Q_ν^m

$$\text{(iii) } \Phi \text{ is finite } \begin{cases} \text{at } r = 0. & \Rightarrow \text{drop } r^{-l-1} \text{ in region } r < r' \\ \text{as } r \to \infty. & \Rightarrow \text{drop } r^{l} \text{ in region } r > r' \end{cases}$$

$$\Rightarrow \Phi = \begin{cases} \sum\limits_{l=0}^{\infty} A_l r^l P_l(\cos\gamma), & r < r' \\ \sum\limits_{l=0}^{\infty} B_l r^{-l-1} P_l(\cos\gamma), & r > r' \end{cases}$$

The formal method to solve for A_l and B_l is to match the b.c. at $r = r'$ (as will be done in Sec. 3.9). Here we obtain A_l and B_l by exploiting the fact that we already know $\Phi = 1/|\mathbf{x}-\mathbf{x}'|$ (for a point source, $q = 4\pi\varepsilon_0$). So, by the uniqueness theorem, we have

$$\Phi = \frac{1}{|\mathbf{x}-\mathbf{x}'|} = \begin{cases} \sum\limits_{l=0}^{\infty} A_l r^l P_l(\cos\gamma) \quad , & r < r' \\ \sum\limits_{l=0}^{\infty} B_l r^{-l-1} P_l(\cos\gamma) \ , & r > r' \end{cases}$$

For $\gamma = 0$, we have $P_l(1) = 1$ and $|\mathbf{x}-\mathbf{x}'| = |r - r'|$. Hence,

$$\frac{1}{|r-r'|} = \begin{cases} \sum\limits_{l=0}^{\infty} A_l r^l, & r < r' \\ \sum\limits_{l=0}^{\infty} B_l r^{-l-1}, & r > r' \end{cases}$$

$$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}y^3 + \cdots$$

But
$$\frac{1}{|r-r'|} = \begin{cases} \dfrac{1}{r'-r} = \dfrac{1}{r'}\dfrac{1}{1-\frac{r}{r'}} \overset{\downarrow}{=} \dfrac{1}{r'}\sum\limits_{l=0}^{\infty}\left(\frac{r}{r'}\right)^l = \sum\limits_{l=0}^{\infty}\dfrac{r^l}{r'^{l+1}}, & r < r' \\ \dfrac{1}{r-r'} = \dfrac{1}{r}\dfrac{1}{1-\frac{r'}{r}} = \dfrac{1}{r}\sum\limits_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l = \sum\limits_{l=0}^{\infty}\dfrac{r'^l}{r^{l+1}}\ , & r > r' \end{cases}$$

Equating the RHS of this equation to the RHS of the equation on the previous page, we obtain

$$A_l = \frac{1}{r'^{l+1}}\ ,\ B_l = r'^l \ \Rightarrow\ \frac{1}{|\mathbf{x}-\mathbf{x}'|} = \begin{cases} \sum\limits_{l=0}^{\infty}\dfrac{r^l}{r'^{l+1}}P_l(\cos\gamma), & r < r' \\ \sum\limits_{l=0}^{\infty}\dfrac{r'^l}{r^{l+1}}P_l(\cos\gamma)\ , & r > r' \end{cases}$$

or
$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\gamma), \qquad \text{[two equations in one]} \qquad (3.38)$$

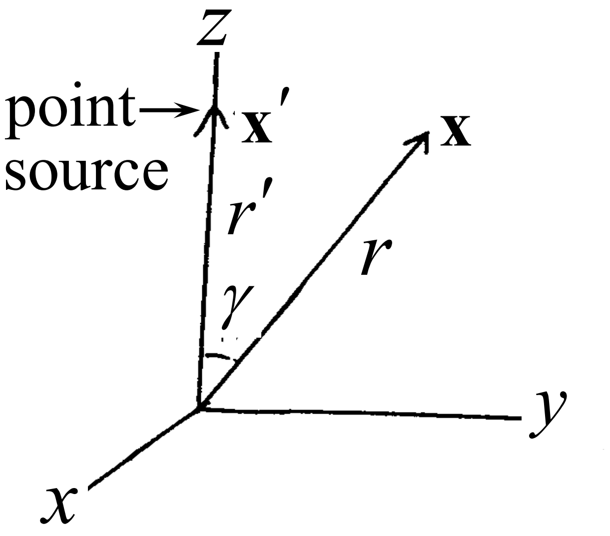
where $r_<$ $(r_>)$ is the smaller (larger) of r and r'.

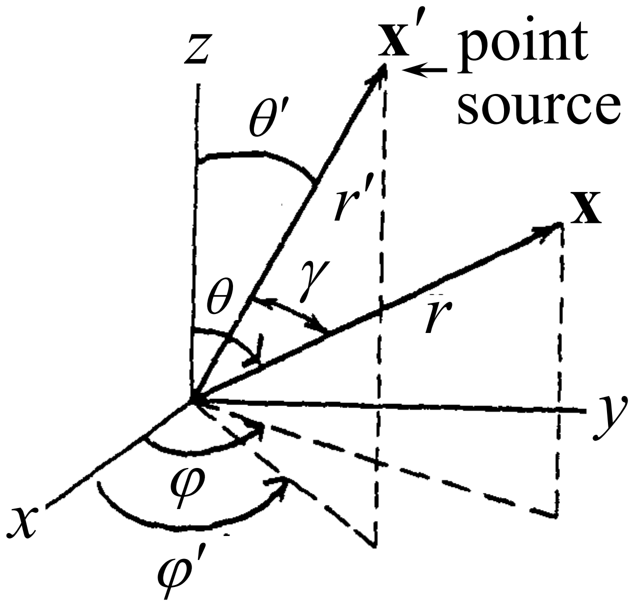
3.3 Boundary-Value Problems with Azimuthal Symmetry (*continued*)

Rewrite (3.38):

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|}=\sum_{l=0}^{\infty}\frac{r_<^{l}}{r_>^{l+1}}P_l(\cos\gamma)$$

This equation was derived with the unit point source located on the z-axis (upper figure). However, it depends only on the magnitudes ($r,\ r'$) of $\mathbf{x}$ and $\mathbf{x}'$ and the angle (γ) between $\mathbf{x}$ and $\mathbf{x}'$. So we expect the expression in (3.38) can be cast into a general form which holds true when the unit point charge is at an arbitrary point (lower figure). We may obtain the general form by way of the addition theorem.

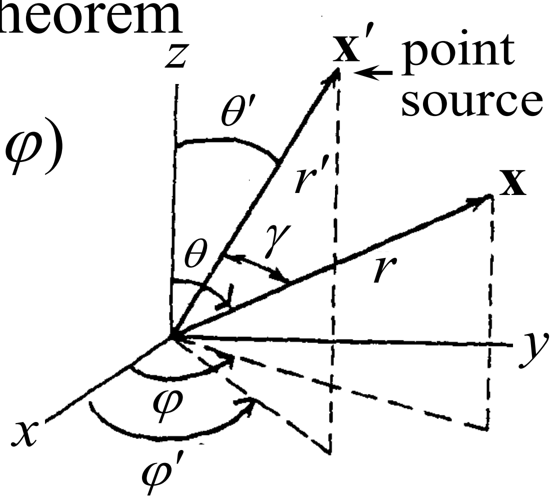




3.3 Boundary-Value Problems with Azimuthal Symmetry (*continued*)

Substituting the RHS of the addition theorem

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi) \qquad (3.62)$$



for $P_l(\cos\gamma)$ in $\dfrac{1}{|\mathbf{x}-\mathbf{x}'|} = \sum_{l=0}^{\infty}\dfrac{r_<^l}{r_>^{l+1}}P_l(\cos\gamma)$, (3.38)

we get $$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi) \qquad (3.70)$$

So, we started with a physics problem (the potential of a point charge in infinite space), but end up with a mathematical relation in (3.70).

Question: Why write a simple function $\Phi = 1/|\mathbf{x}-\mathbf{x}'|$ in such a complicated form? (See next problem.)

3.3 Boundary-Value Problems with Azimuthal Symmetry (*continued*)

Example 4: Potential due to charge Q uniformly distributed on a circular ring of radius a.

Sol:

Let $\rho(\mathbf{x}) = K\delta(\theta-\alpha)\delta(r-c)$ in spherical coordinates

$$Q = \int \rho(\mathbf{x})d^3x$$

$$= K\int \delta(\theta-\alpha)\delta(r-c)\overbrace{r^2 \sin\theta dr d\theta d\varphi}^{d^3x}$$

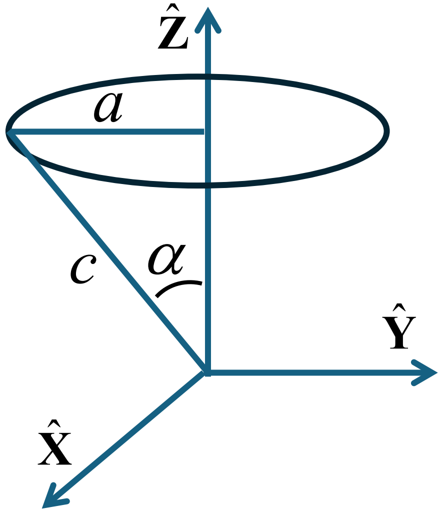
$$= 2\pi K c^2 \sin\alpha$$

$$\Rightarrow \quad K = \frac{Q}{2\pi c^2 \sin\alpha}$$

$$\Rightarrow \rho(\mathbf{x}) = \frac{Q}{2\pi c^2 \sin\alpha}\delta(\theta-\alpha)\delta(r-c)$$

$$\underset{\uparrow}{=} \frac{Q}{2\pi c^2}\delta(\cos\theta-\cos\alpha)\delta(r-c)$$

$$\boxed{\delta[f(x)] = \frac{\delta(x-a)}{|f'(a)|}}$$



$$\rho(\mathbf{x}')=\frac{Q}{2\pi c^2}\delta(\cos\theta'-\cos\alpha)\delta(r'-c)$$

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|}=4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi)$$

$$\Phi(\mathbf{x})=\frac{1}{4\pi\varepsilon_0}\int_V\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' \longleftarrow$$

$$=\frac{Q}{2\pi\varepsilon_0 c^2}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\int_V r'^2dr'd\cos\theta'd\varphi'\left[\begin{array}{l}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi)\\ \cdot\delta(\cos\theta'-\cos\alpha)\delta(r'-c)\end{array}\right]$$

$$Y_{lm}(\theta',\varphi')=\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta')e^{im\varphi'}$$

Apparently, only the $m=0$ terms survive the φ' integration.

$$\Rightarrow\Phi(\mathbf{x})=\frac{Q}{4\pi\varepsilon_0 c^2}\sum_{l=0}^{\infty}\int_0^{\infty}r'^2dr'\int_{-1}^{1}d\cos\theta'\left[\begin{array}{l}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\theta')P_l(\cos\theta)\\ \cdot\delta(\cos\theta'-\cos\alpha)\delta(r'-c)\end{array}\right]$$

$$=\frac{Q}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\alpha)P_l(\cos\theta)$$

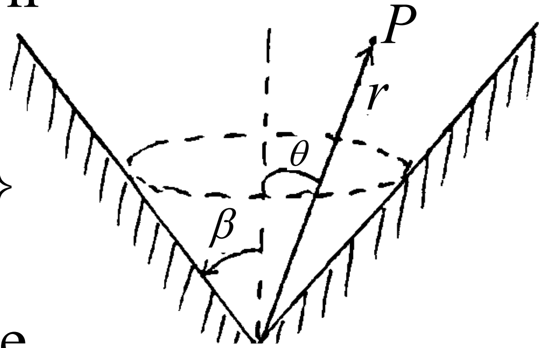
Jackson uses a slightly different method to derive this. See p.103.

Self-learning

3.4 Behavior of Fields in a Conical Hole or Near a Sharp Point

Consider the source-free configurations shown in the figures.

$$\nabla^2\Phi = 0 \Rightarrow \Phi = \begin{Bmatrix} r^{\nu} \\ r^{-\nu-1} \end{Bmatrix} \begin{Bmatrix} P_\nu^m(\cos\theta) \\ Q_\nu^m(\cos\theta) \end{Bmatrix} \begin{Bmatrix} e^{im\varphi} \\ e^{-im\varphi} \end{Bmatrix}$$

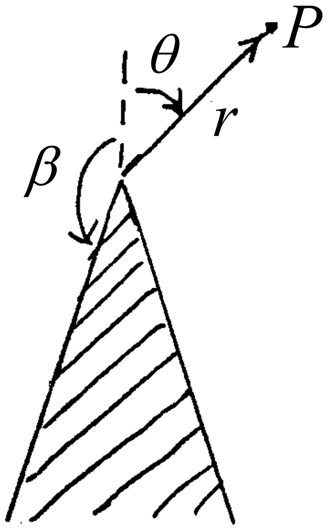


(i) The geometry is indep. of φ (We also assume that the b.c. is indep. of φ.) $\Rightarrow\ m = 0$

(ii) $Q_\nu^m(\cos\theta)$ diverges at $\theta = 0$ or $\cos\theta = 1$.

$\Rightarrow$ drop $Q_\nu^m(\cos\theta)$

Hence, $\Phi = \begin{Bmatrix} r^{\nu} \\ r^{-\nu-1} \end{Bmatrix} P_\nu(\cos\theta)$



Note: $P_\nu(x)$ diverges at $x = -1$ unless $\nu =$ integer. However, in this problem, we have $0 \le \beta < \pi \ \Rightarrow\ \cos\theta \ne -1$ in the region of interest. Hence, ν is not required to be an integer.

Rewrite: $\Phi = \begin{Bmatrix} r^{\nu} \\ r^{-\nu-1} \end{Bmatrix} P_{\nu}(\cos\theta)$

(iii) We are interested in the general behavior of the potential and fields in the neighborhood of $r = 0$ and not in the full solution with specified boundary condition imposed at large r.

$$\Phi = \sum_{k=1}^{\infty} A_k r^{\nu_k} P_{\nu_k}(\cos\theta) \tag{3.44}$$

(iv) $\Phi = 0$ at $\theta = \beta \Rightarrow P_{\nu}(\cos\beta) = 0 \;\Rightarrow\; \nu = \nu_1,\ \nu_2,\ \nu_3, \ldots (\nu > 0)$

Note: In the boundary condition: $P_{\nu}(\cos\beta) = 0$, β is fixed and ν is the eigenvalue to be solved.

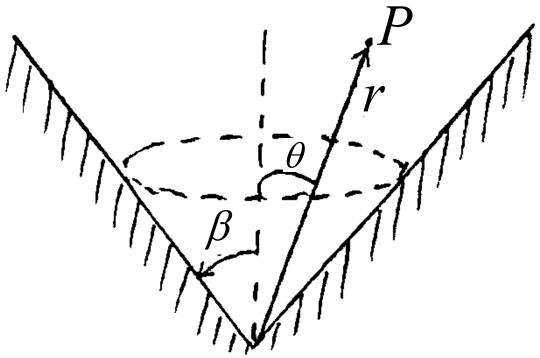
$$\Rightarrow \Phi \simeq A_1 r^{\nu_1} P_{\nu_1}(\cos\theta), \tag{3.45}$$

where ν_1 is the smallest eigenvalue [the first root of $P_{\nu}(\cos\beta) = 0$].

Self-learning

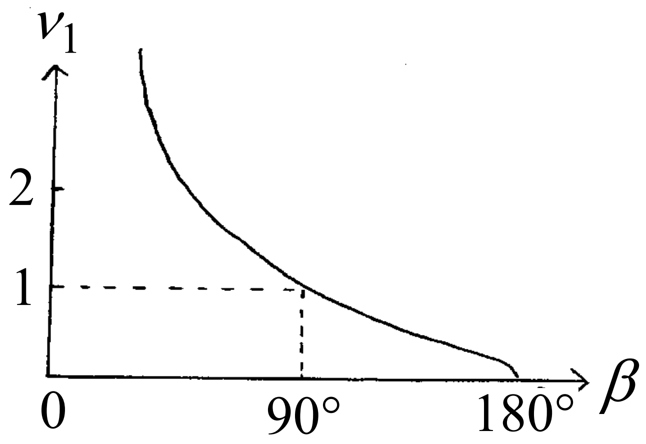
3.4 Behavior of Fields in a Conical Hole or Near a Sharp Point (*continued*)

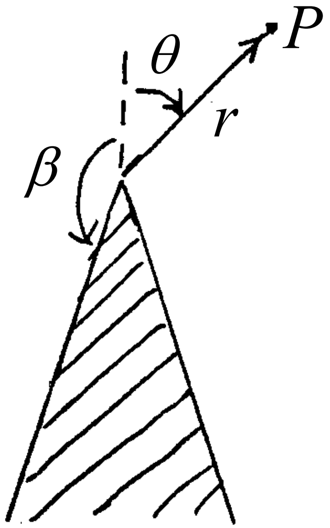
$$\Rightarrow \begin{cases} E_r = -\dfrac{\partial \phi}{\partial r} \overset{\boxed{r \to 0}}{\approx} -\nu_1 A_1 r^{\nu_1 - 1} P_{\nu_1}(\cos\theta) \propto r^{\nu_1 - 1} \\ E_\theta = -\dfrac{1}{r}\dfrac{\partial \phi}{\partial \theta} \approx A_1 r^{\nu_1 - 1} \sin\theta P'_{\nu_1}(\cos\theta) \propto r^{\nu_1 - 1} \\ \sigma = -\varepsilon_0 E_\theta(\theta = \beta) \approx -A_1 \varepsilon_0 r^{\nu_1 - 1} \sin\beta P'_{\nu_1}(\cos\beta) \propto r^{\nu_1 - 1} \end{cases}$$



Behavior of ν_1 as a function of β is shown in the figure below. Note that

$$\begin{cases} \nu_1 > 1, \text{ if } \beta < 90^\circ \\ \nu_1 = 1, \text{ if } \beta = 90^\circ \\ \nu_1 < 1, \text{ if } \beta > 90^\circ \end{cases}$$





When $\beta < 90^{\text{o}}$ (conical hole), both E and $\sigma \to 0$ as $r \to 0$.

$\beta = 170^\circ \;\Rightarrow\; \nu_1 = 0.2;$ $\beta = 179^\circ \;\Rightarrow\; \nu_1 = 0.1$

3.4 Behavior of Fields in a Conical Hole or Near a Sharp Point (*continued*)

However, when $\beta > 90^{\mathrm{o}}$ (sharp point), both E and $\sigma \to \infty$ as $r \to 0$. Large electric field ($E > 2.5 \times 10^4$ V/cm) can cause the air to breakdown and form a conducting path in the air for the sharp point to discharge. This is the principle of the lightning rod (pp. 77-78.)

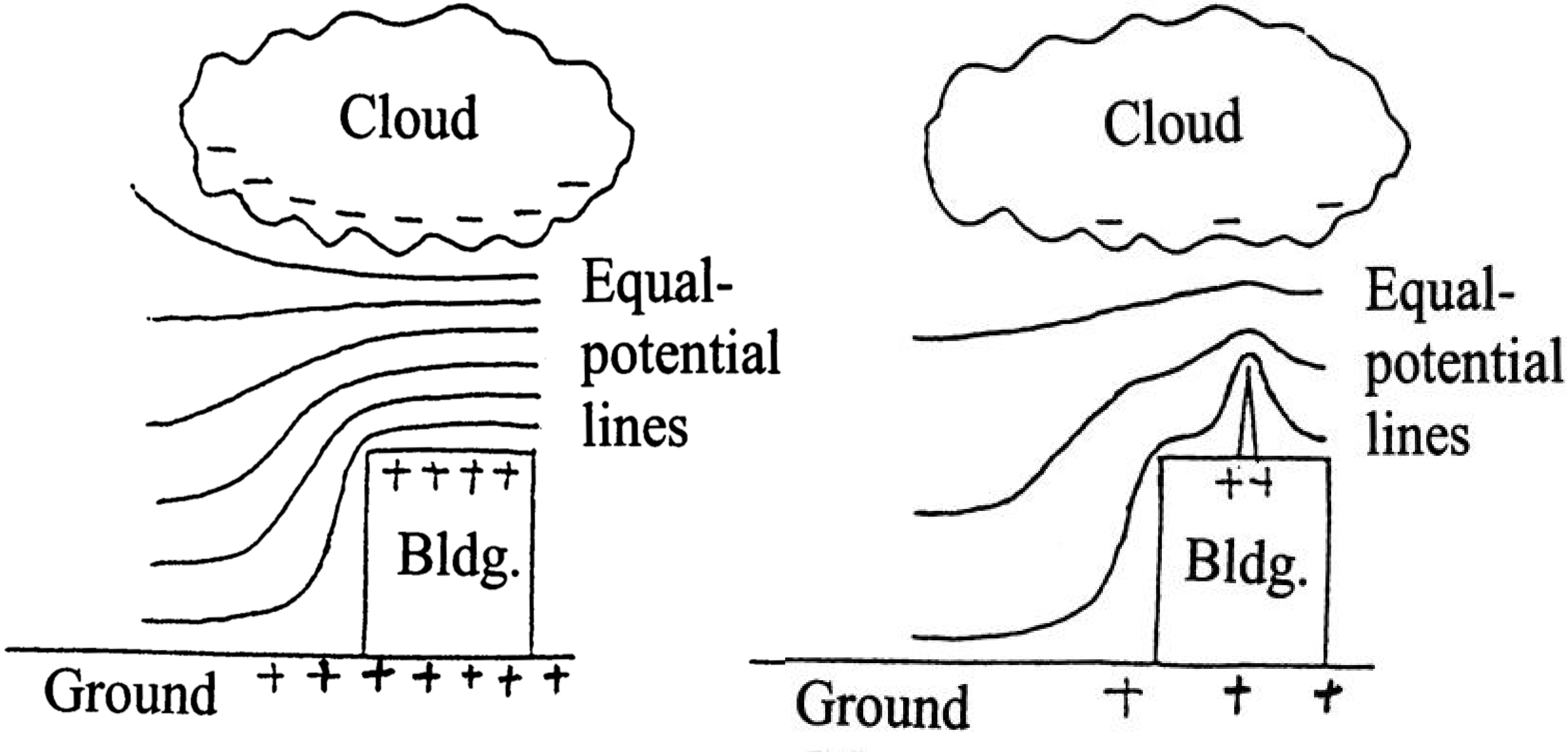
If the region of interest is bounded by the surface at $r = a$, the coefficients A_k in (3.44) can be determined by the b.c. $\Phi(r = a) = \Phi(\theta)$ through

$$\Phi(\theta) = \sum_{k=1}^{\infty} A_k a^{\nu_k} P_{\nu_k}(\cos\theta)$$

If $\Phi(r = a) = \Phi(\theta) = 0,$ then all $A_k = 0. \Rightarrow \Phi = 0$ everywhere

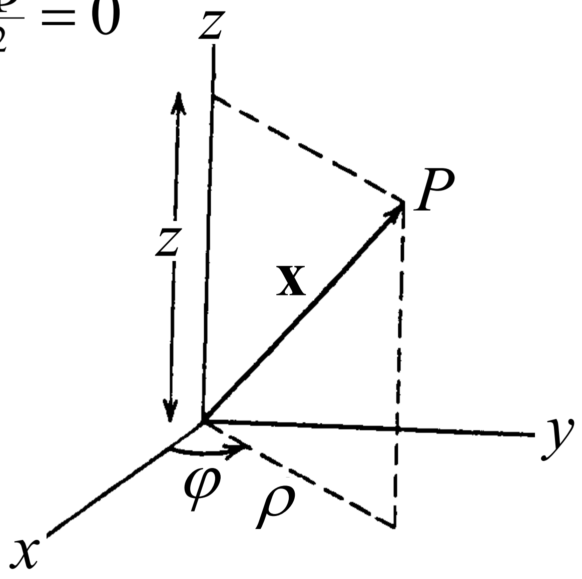
In reality, the lightning rod is not perfectly sharp. Hence, Φ is finite at the tip, and on a clear day when there is a small potential difference between the ground the clouds, the lightning rod will not discharge.

A physical picture of the lightning rod



3.7 Laplace Equation in Cylindrical Coordinates; Bessel Functions

$$\nabla^2\Phi(\mathbf{x})=0\Rightarrow\frac{\partial^2\Phi}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial\Phi}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\varphi^2}+\frac{\partial^2\Phi}{\partial z^2}=0$$



Let $\Phi(\mathrm{x})=R(\rho)Q(\varphi)Z(z)$

$$\Rightarrow\begin{cases}\dfrac{\partial^2 Z}{\partial z^2}-k^2Z=0\Rightarrow Z=e^{\pm kz}\\[2ex]\dfrac{\partial^2 Q}{\partial\varphi^2}+\nu^2Q=0\Rightarrow Q=e^{\pm i\nu\varphi}\\[2ex]\dfrac{\partial^2 R}{\partial\rho^2}+\dfrac{1}{\rho}\dfrac{\partial R}{\partial\rho}+\left(k^2-\dfrac{\nu^2}{\rho^2}\right)R=0\Rightarrow R=J_\nu(k\rho),\ N_\nu(k\rho)\end{cases}$$

where J_ν and N_ν are Bessel functions of the first and second kind, respectively (see following pages).

$$\Rightarrow\Phi=\begin{Bmatrix}J_\nu(k\rho)\\N_\nu(k\rho)\end{Bmatrix}\begin{Bmatrix}e^{i\nu\varphi}\\e^{-i\nu\varphi}\end{Bmatrix}\begin{Bmatrix}e^{kz}\\e^{-kz}\end{Bmatrix}\qquad(3)$$

Bessel Functions : If we let $x = k\rho$, the equation for R takes the standard form of the Bessel equation,

$$\frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} + \left(1 - \frac{\nu^2}{x^2}\right)R = 0 \qquad (3.77)$$

with solutions $J_\nu(x)$ and $N_\nu(x)$, from which we define the Hankel functions:

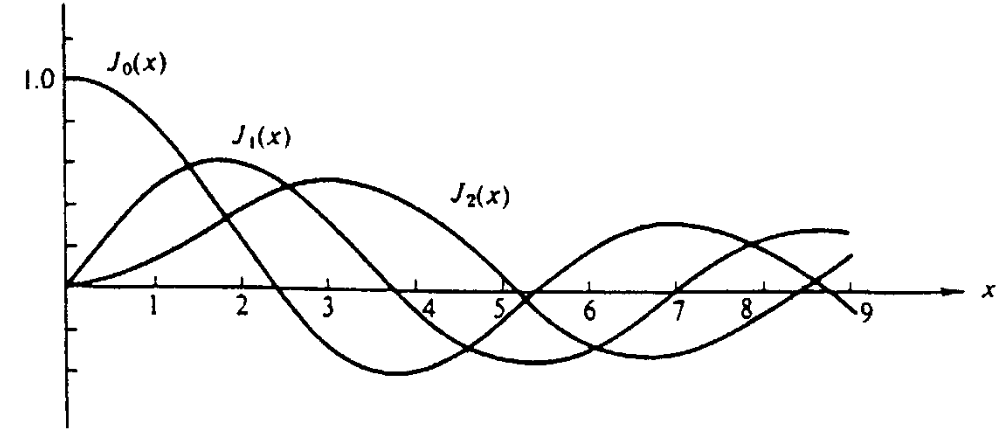
$$\begin{cases} H_\nu^{(1)}(x) = J_\nu(x) + iN_\nu(x) \\ H_\nu^{(2)}(x) = J_\nu(x) - iN_\nu(x) \end{cases} \qquad (3.86)$$

and the modified Bessel functions (Bessel functions of imaginary argument) $\frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} - \left(1 + \frac{\nu^2}{x^2}\right)R = 0$

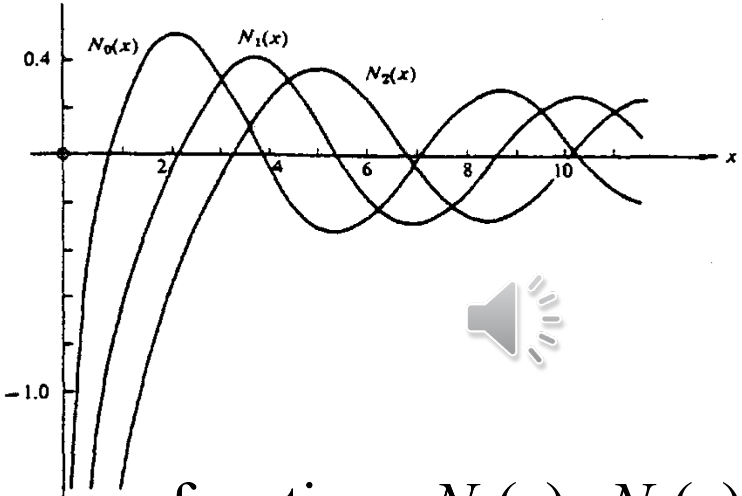
$$\begin{cases} I_\nu(x) = i^{-\nu}J_\nu(ix) & (3.100) \\ K_\nu(x) = \frac{\pi}{2}i^{\nu+1}H_\nu^{(1)}(ix) & (3.101) \end{cases}$$

See Jackson pp. 112-116, Gradshteyn & Ryzhik, and Abramowitz & Stegun for properties of these special functions.

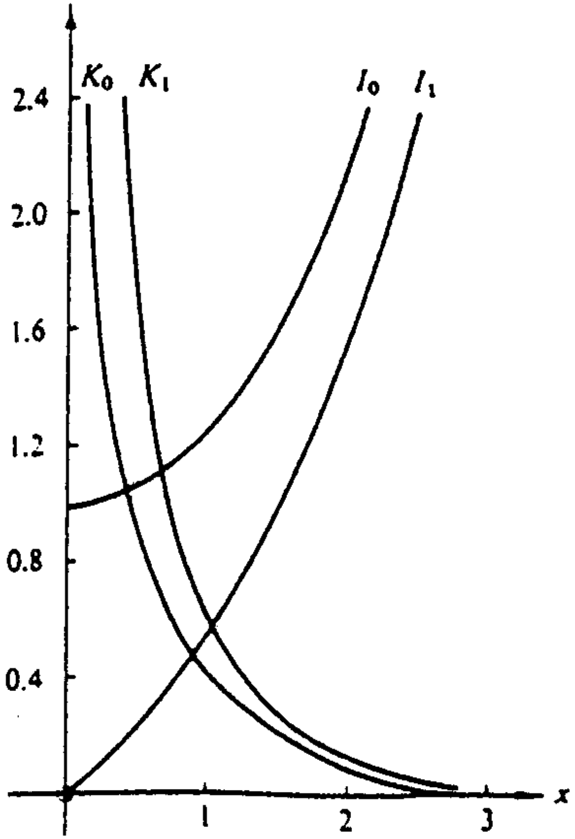
3.7 Laplace Equation in Cylindrical Coordinates; Bessel Functions (*continued*)



Bessel functions $J_0(x)$, $J_1(x)$, and $J_2(x)$



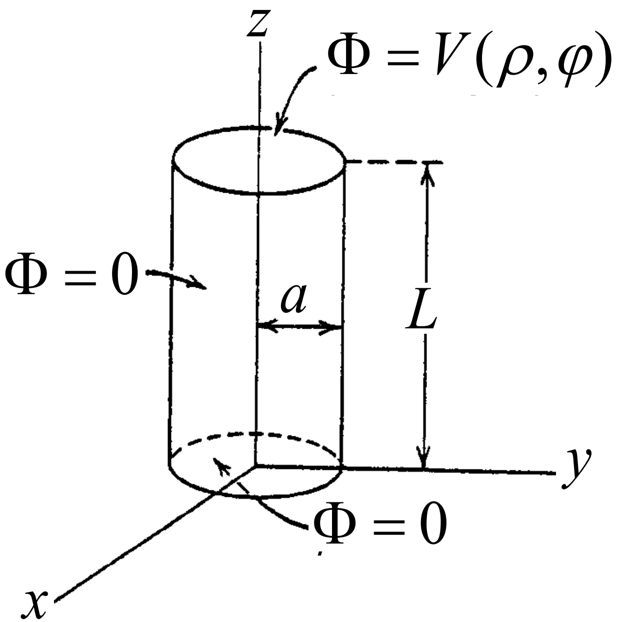
Neumann functions $N_0(x)$, $N_1(x)$, and $N_2(x)$



Modified Bessel functions $I_0(x)$, $I_1(x)$, $K_0(x)$, and $K_1(x)$

3.8 Boundary-Value Problems in Cylindrical Coordinates

Example 1: Potential inside a charge-free cylinder (see figure) with the b.c. $\Phi(z=L)=V(\rho,\varphi)$ and $\Phi=0$ on other surfaces.



Sol:

$$\nabla^2\Phi(\mathbf{x})=0 \Rightarrow \Phi = \begin{Bmatrix} J_\nu(k\rho) \\ N_\nu(k\rho) \end{Bmatrix} \begin{Bmatrix} e^{i\nu\varphi} \\ e^{-i\nu\varphi} \end{Bmatrix} \begin{Bmatrix} e^{kz} \\ e^{-kz} \end{Bmatrix}$$

(i) $Z(z) = Ae^{kz} + Be^{-kz}$

$\Phi = 0$ at $z = 0 \ \Rightarrow Z(0) = 0 \Rightarrow B = -A$

$\Rightarrow Z(z) = A\left(e^{kz} - e^{-kz}\right) = A' \sinh kz$

(ii) $\Phi(\varphi) = \Phi(\varphi + 2\pi)$, i.e. Φ is single-valued.

$\Rightarrow \nu = m =$ integer

$$\Rightarrow Q(\varphi) = \sum_{m=-\infty}^{\infty} C_m e^{im\varphi} = \sum_{m=0}^{\infty} \left(A_m \sin m\varphi + B_m \cos m\varphi\right)$$

(iii) Φ is finite at $\rho = 0$. $\Rightarrow$ drop $N_m(k\rho) \Rightarrow R = J_m(k\rho)$

3.8 Bounday-Value Problems in Cylindrical Coordinates (*continued*)

Rewrite: $R = J_m(k\rho)$

(iv) $\Phi = 0$ at $\rho = a \;\Rightarrow J_m(ka) = 0 \Rightarrow k = k_{mn} = \frac{x_{mn}}{a},\; n = 1,\; 2,\; 3\ldots$

where x_{mn} is the n-th root of $J_m(x) = 0$. Thus,

$$\Phi(\rho,\varphi,z) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(k_{mn}\rho)\sinh(k_{mn}z)(A_{mn}\sin m\varphi + B_{mn}\cos m\varphi)$$

With k fixed by the boundary condition to a set of dicrete values (k_{mn}), we may introduce two properties of $J_m(k_{mn}\rho)$:

The set $J_m(k_{mn}\rho)$ is orthogonal in index n: [m : a fixed number.]

$$\int_0^a J_m(k_{mn'}\rho)J_m(k_{mn}\rho)\rho d\rho = \frac{a^2}{2}[J_{m+1}(k_{mn}a)]^2\delta_{n'n} \qquad (3.95)$$

The set $J_m(k_{mn}x)$ is complete in index n. Hence, any function $f(x)$ can be expanded as $f(x) = \sum_{n=1}^{\infty} C_n J_m(k_{mn}x)$

Questions: (See last page of Appendix A.)

1. Why is $J_m(k_{mn}x)$ orhtogonal and complete in index n instead of m?
2. Why is there a factor ρ in the integrand of (3.95), but not in (3.52)?

Rewrite:

$$\Phi(\rho,\varphi,z)=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}J_m(k_{mn}\rho)\sinh(k_{mn}z)(A_{mn}\sin m\varphi+B_{mn}\cos m\varphi)$$

(v) $\Phi(\rho,\varphi,z=L)=V(\rho,\varphi)$

$$\Rightarrow V(\rho,\varphi)=\sum_{m,n}\sinh(k_{mn}L)J_m(k_{mn}\rho)(A_{mn}\sin m\varphi+B_{mn}\cos m\varphi)$$

Operating both sides with $\int_0^{2\pi}d\varphi\int_0^a\rho d\rho J_m(k_{mn}\rho)\begin{Bmatrix}\sin m\varphi\\ \cos m\varphi\end{Bmatrix}$ and

making use of the orthogonal properties of $\sin m\varphi$ and $\cos m\varphi$, and

the relation: $\int_0^a J_m(k_{mn'}\rho)J_m(k_{mn}\rho)\rho d\rho=\frac{a^2}{2}[J_{m+1}(k_{mn}a)]^2\delta_{n'n}$ (3.95)

$$\Rightarrow\begin{Bmatrix}A_{mn}\\ B_{mn}\end{Bmatrix}=\frac{2\operatorname{cosech}(k_{mn}L)}{\pi a^2J_{m+1}^2(k_{mn}a)}\int_0^{2\pi}d\varphi\int_0^a\rho d\rho V(\rho,\varphi)J_m(k_{mn}\rho)\begin{Bmatrix}\sin m\varphi\\ \cos m\varphi\end{Bmatrix}$$

(for $m=0$, use $\frac{1}{2}B_{0n}$)

Example 2: Potential in the charge-free semi-infinite space $z \geq 0$

subject to the b.c. $\begin{cases} \Phi(\rho, \varphi, z=0) = V(\rho, \varphi) \\ \Phi(\rho \to \infty, \varphi, z) = 0 \end{cases}$

Sol:

$$\nabla^2 \Phi(\mathbf{x}) = 0 \Rightarrow \Phi = \begin{Bmatrix} J_\nu(k\rho) \\ N_\nu(k\rho) \end{Bmatrix} \begin{Bmatrix} e^{i\nu\varphi} \\ e^{-i\nu\varphi} \end{Bmatrix} \begin{Bmatrix} e^{kz} \\ e^{-kz} \end{Bmatrix}$$

(i) Φ remains finite as $z \to \infty$. $\Rightarrow$ drop e^{kz} $\Rightarrow Z(z) = Ae^{-kz}$

(ii) $\Phi(\varphi) = \Phi(\varphi + 2\pi)$ $\Rightarrow \nu = m =$ integer

$$\Rightarrow Q(\varphi) = \sum_{m=0}^{\infty} \left(A_m \sin m\varphi + B_m \cos m\varphi \right)$$

(iii) Φ is finite at $\rho = 0$. $\Rightarrow$ drop $N_m(k\rho) \Rightarrow R = J_m(k\rho)$

(iv) $\Phi = 0$ at $\rho \to \infty$ $\Rightarrow J_m(k \cdot \infty) = 0$ $\Rightarrow$ continuous eigenvalue k

$$\Rightarrow \Phi(\rho, \varphi, z) = \sum_{m=0}^{\infty} \int_0^\infty dk e^{-kz} J_m(k\rho) \left[A_m(k) \sin m\varphi + B_m(k) \cos m\varphi \right] \tag{3.106}$$

Rewrite (3.106) with variable k changed to k':

$$\Phi(\rho,\varphi,z)=\sum_{m=0}^{\infty}\int_0^{\infty}dk'e^{-k'z}J_m(k'\rho)\left[A_m(k')\sin m\varphi+B_m(k')\cos m\varphi\right]$$

(v) $\Phi(\rho,\varphi,z=0)=V(\rho,\varphi)$

$$\Rightarrow\ V(\rho,\varphi)=\sum_{m=0}^{\infty}\int_0^{\infty}dk'J_m(k'\rho)\left[A_m(k')\sin m\varphi+B_m(k')\cos m\varphi\right]$$

Operating both sides with $\int_0^{2\pi}d\varphi\int_0^{\infty}\rho d\rho J_m(k\rho)\begin{Bmatrix}\sin m\varphi\\ \cos m\varphi\end{Bmatrix}$ and making use of the orthogonal properties of $\sin m\varphi$ and $\cos m\varphi$, and the relation: $\int_0^{\infty}xJ_m(kx)J_m(k'x)dx=\frac{1}{k}\delta(k-k')$ (3.108)

$$\Rightarrow\ \begin{Bmatrix}A_m(k)\\ B_m(k)\end{Bmatrix}=\frac{k}{\pi}\int_0^{2\pi}d\varphi\int_0^{\infty}\rho d\rho V(\rho,\varphi)J_m(k\rho)\begin{Bmatrix}\sin m\varphi\\ \cos m\varphi\end{Bmatrix}\qquad(3.109)$$

For $m=0$, use $\frac{1}{2}B_0(k)$ in series (3.106).

3.9 Expansion of Green Functions in Spherical Coordinates

The Green function for an electrostatic potential problem satisfies

$$\nabla^2 G(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}')$$

with $G(\mathbf{x},\mathbf{x}') = 0$ for $\mathbf{x}$ on the boundary surface.

Question: Jackson p.120 states the b.c. as "$G(\mathbf{x}, \mathbf{x}') = 0$ for either $\mathbf{x}$ or $\mathbf{x}'$ on the boundary surface." Why?

Case 1: Green function in infinite space

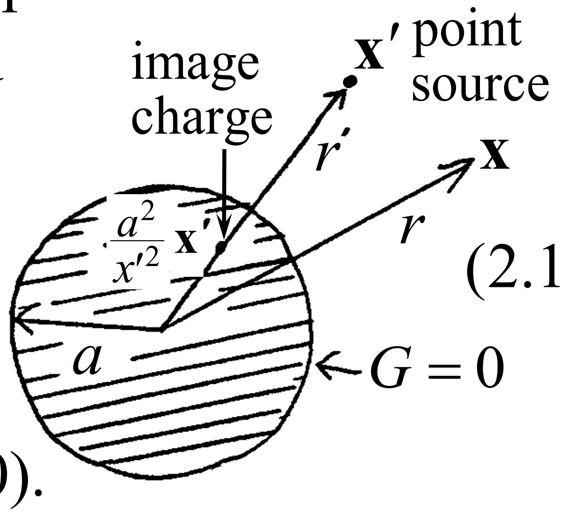
The simplest form is $G(\mathbf{x},\mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|}$ (Sec. 1.10).

It can be expressed as an expansion in spherical coordinates as (Sec. 3.7)

$$G(\mathbf{x},\mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\theta',\varphi') Y_{lm}(\theta,\varphi) \qquad (3.70)$$

Case 2 : Green function outside a conducting sphere

By the method of images, we have obtained the Green function in Sec. 2.6,



$$G(\mathbf{x},\mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|} - \frac{a}{x'\left|\mathbf{x}-\frac{a^2}{x'^2}\mathbf{x}'\right|} \qquad (2.16)$$

The first term in (2.16) is expanded in (3.70).

The second term can be expanded using (3.70). Since $|\mathbf{x}| > \left|\frac{a^2}{x'^2}\mathbf{x}'\right|$,

we substitute $r_> = |\mathbf{x}| = r$ and $r_< = \left|\frac{a^2}{x'^2}\mathbf{x}'\right| = \frac{a^2}{r'}$ into (3.70) to obtain

$$\frac{a}{\underset{\nearrow r'}{x'}\left|\mathbf{x}-\frac{a^2}{x'^2}\mathbf{x}'\right|} = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{a\left(\frac{a^2}{r'}\right)^l}{r'r^{l+1}}Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi)$$

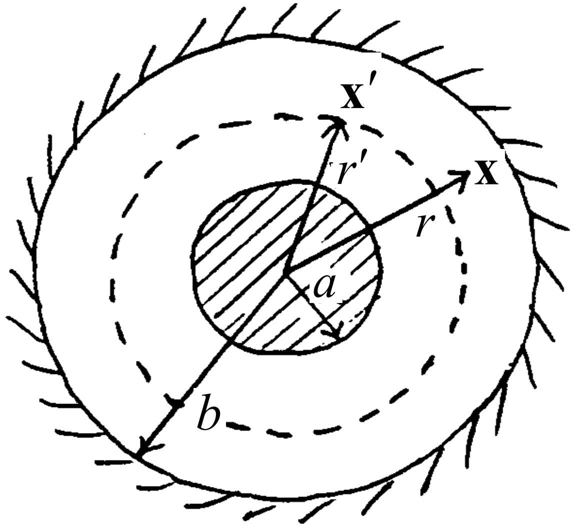
$$\Rightarrow G(\mathbf{x},\mathbf{x}') = 4\pi\sum_{l,m}\frac{1}{2l+1}\left[\frac{r_<^l}{r_>^{l+1}} - \frac{1}{a}\left(\frac{a^2}{rr'}\right)^{l+1}\right]Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi), \quad (3.114)$$

Self-learning

Case 3: Green function inside a spherical shell bounded by grounded conductors (see figure)

$$\nabla^2 G(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}')$$

with the b.c. $G(r=a)=G(r=b)=0$



This problem is difficult to solve by the method of images. We will solve it by a systematic method: **method of expansion**.

Write $\delta(\mathbf{x}-\mathbf{x}')$ in spherical coordinates,

$$\delta(\mathbf{x}-\mathbf{x}') = \frac{1}{r^2}\delta(r-r')\delta(\varphi-\varphi')\delta(\cos\theta-\cos\theta')$$

Use the completeness relation (3.56) for $\delta(\varphi-\varphi')\delta(\cos\theta-\cos\theta')$

$$\Rightarrow \quad \delta(\mathbf{x}-\mathbf{x}') = \frac{1}{r^2}\delta(r-r')\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi) \qquad (3.117)$$

Note that, in (3.117), we have decomposed a point charge into an infinite number of spherical "charge layers", all of radius r'.

Self-learning

3.9 Expansion of Green Functions in Spherical Coordinates *(continued)*

$$\Rightarrow \nabla^2 G(\mathbf{x},\mathbf{x}') = -4\pi \frac{1}{r^2}\delta(\underset{\substack{\uparrow\\ \boxed{\text{variable}}}}{r} - \underset{\substack{\uparrow\\ \boxed{\text{constant}}}}{r'}) \sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y^*_{lm}\underbrace{(\theta',\varphi')}_{\boxed{\text{constants}}} Y_{lm}\underbrace{(\theta,\varphi)}_{\boxed{\text{variables}}} \qquad (4)$$

The RHS of this equation is an expansion in spherical harmonics, which suggests that we expand $G(\mathbf{x},\mathbf{x}')$ similarly. This is possible since $Y_{lm}(\theta,\varphi)$ form a complete set.

$$G(\mathbf{x},\mathbf{x}') = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{lm}(\underset{\substack{\uparrow\\ \boxed{\text{variable}}}}{r} \,|\, \underbrace{r',\theta',\varphi'}_{\boxed{\text{constants}}}) Y_{lm}\underbrace{(\theta,\varphi)}_{\boxed{\text{variables}}}, \qquad (3.118)$$

where A_{lm} is a function of r to be solved from (4).

Expressing A_{lm} as

$$A_{lm}(r\,|\,r',\theta',\varphi') = g_l(r,r')Y^*_{lm}(\theta',\varphi') \qquad (5)$$

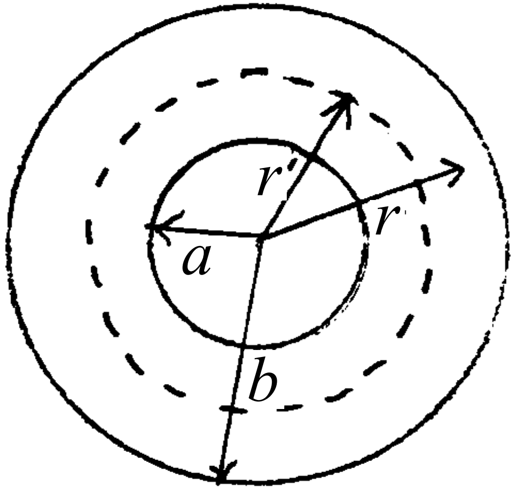
and sub. (5) into (4), we get the equation for $g_l(r,r')$ (see Sec. 3.1),

$$\frac{1}{r}\frac{d^2}{dr^2}[rg_l(r,r')]-\frac{l(l+1)}{r^2}g_l(r,r')=-\frac{4\pi}{r^2}\delta(r-r') \qquad (3.120)$$

Divide the space into $r<r'$ and $r>r'$. In each region, (3.120) reduces to

$$\frac{1}{r}\frac{d^2}{dr^2}[rg_l(r,r')]-\frac{l(l+1)}{r^2}g_l(r,r')=0$$

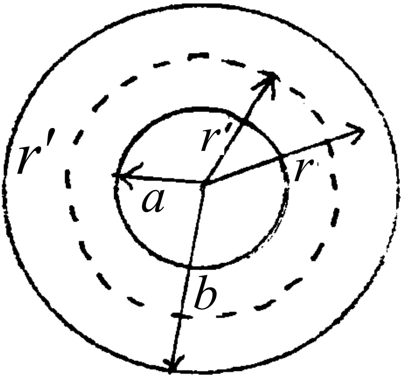
$$\Rightarrow \quad g_l(r,r')=\begin{cases} Ar^l+Br^{-l-1}, & r<r' \\ A'r^l+B'r^{-l-1}, & r>r' \end{cases} \qquad (6)$$



The remaining job is to find 4 boundary conditions to solve for the 4 constants $A,\ B,\ A'$, and B' in (6).

Self-learning

3.9 Expansion of Green Functions in Spherical Coordinates (*continued*)



(i) $g_l(r=a,r')=0 \;\Rightarrow g_l(r,r')=A\left(r^l-\dfrac{a^{2l+1}}{r^{l+1}}\right),\quad r<r'$

(ii) $g_l(r=b,r')=0 \;\Rightarrow g_l(r,r')=B'\left(\dfrac{1}{r^{l+1}}-\dfrac{r^l}{b^{2l+1}}\right),\quad r>r'$

(iii) $g_l(r,r')$ is continuous at $r=r'$.

Physical reason: ϕ is continuous across the charge layer at $r=r'$. (E is finite at $r=r'$. $\Rightarrow \Delta\phi=\lim\limits_{\Delta r\to 0}E\Delta r=0$). Thus,

$$A\left(r'^l-\frac{a^{2l+1}}{r'^{l+1}}\right)=B'\left(\frac{1}{r'^{l+1}}-\frac{r'^l}{b^{2l+1}}\right)\Rightarrow \frac{A}{B'}=\frac{\dfrac{1}{r'^{l+1}}-\dfrac{r'^l}{b^{2l+1}}}{r'^l-\dfrac{a^{2l+1}}{r'^{l+1}}}\Rightarrow\begin{cases}A=C\left(\dfrac{1}{r'^{l+1}}-\dfrac{r'^l}{b^{2l+1}}\right)\\[2ex] B'=C\left(r'^l-\dfrac{a^{2l+1}}{r'^{l+1}}\right)\end{cases}$$

$$\Rightarrow g_l(r,r')=\begin{cases}C\left(\dfrac{1}{r'^{l+1}}-\dfrac{r'^l}{b^{2l+1}}\right)\left(r^l-\dfrac{a^{2l+1}}{r^{l+1}}\right), & r<r'\\[2ex] C\left(r'^l-\dfrac{a^{2l+1}}{r'^{l+1}}\right)\left(\dfrac{1}{r^{l+1}}-\dfrac{r^l}{b^{2l+1}}\right), & r>r'\end{cases}$$

$$=C\left(r_<^l-\frac{a^{2l+1}}{r_<^{l+1}}\right)\left(\frac{1}{r_>^{l+1}}-\frac{r_>^l}{b^{2l+1}}\right)\qquad(3.122)$$

Self-learning

3.9 Expansion of Green Functions in Spherical Coordinates (*continued*)

Rewrite (3.120): $\frac{1}{r}\frac{d^2}{dr^2}[rg_l(r,r')] - \frac{l(l+1)}{r^2} g_l(r,r') = -\frac{4\pi}{r^2}\delta(r-r')$

(iv) We need one more condition to get the remaining constant C in (3.122). Physically, this condition is related to the discontinuity of $E_r (\propto \frac{d}{dr} g_l)$ across the charge layer at $r = r'$. Mathematicaly, we integrate the delta funtion in (3.120) to bring out the discontinuity. Multiply (3.120) by r and integrate from $r' - \varepsilon$ to $r' + \varepsilon$ $(\varepsilon \to 0)$

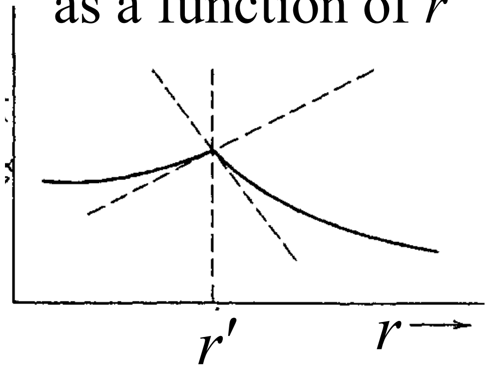
$$\Rightarrow \frac{d}{dr}[rg_l(r,r')]_{r'+\varepsilon} - \frac{d}{dr}[rg_l(r,r')]_{r'-\varepsilon} = -\frac{4\pi}{r'}$$

$$\Rightarrow -\tfrac{C}{r'}\left[1-(\tfrac{a}{r'})^{2l+1}\right]\left[l+(l+1)(\tfrac{r'}{b})^{2l+1}\right]$$

$$-\tfrac{C}{r'}\left[(l+1)+l(\tfrac{a}{r'})^{2l+1}\right]\left[1-(\tfrac{r'}{b})^{2l+1}\right] = -\frac{4\pi}{r'}$$

$$\Rightarrow C = \frac{4\pi}{(2l+1)\left[1-(\tfrac{a}{b})^{2l+1}\right]}$$

rg_l (solid line) as a function of r



Substituting C into (3.122), we get

$$G(\mathbf{x},\mathbf{x}')$$

$$=4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{Y_{lm}^{*}(\theta',\varphi')Y_{lm}(\theta,\varphi)}{(2l+1)\left[1-(\frac{a}{b})^{2l+1}\right]}\left(r_<^l-\frac{a^{2l+1}}{r_<^{l+1}}\right)\left(\frac{1}{r_>^{l+1}}-\frac{r_>^l}{b^{2l+1}}\right)\qquad(3.125)$$

Limiting case 1: $a\to 0\ \&\ b\to\infty$, $(3.125)\Rightarrow(3.70)$

$$G(\mathbf{x},\mathbf{x}')=\frac{1}{|\mathbf{x}-\mathbf{x}'|}=4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^{*}(\theta',\varphi')Y_{lm}(\theta,\varphi)\qquad(3.70)$$

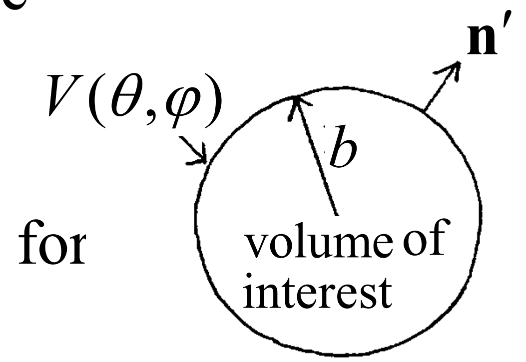
Limiting case 2: $b\to\infty$, $(3.125)\Rightarrow(3.114)$

$$G(\mathbf{x},\mathbf{x}')=4\pi\sum_{l,m}\frac{1}{2l+1}\left[\frac{r_<^l}{r_>^{l+1}}-\frac{1}{a}\left(\frac{a^2}{rr'}\right)^{l+1}\right]Y_{lm}^{*}(\theta',\varphi')Y_{lm}(\theta,\varphi),\quad(3.114)$$

3.10 Solution of Potential Problems with the Spherical Green Function Expansion

Example 1: Potential inside a charge-free sphere of radius b subject to the b.c.

$$\Phi(r=b)=V(\theta,\varphi)$$



Sol: Since we already have the Green function for this problem, it is convenient to use the formal solution derived in Sec. 1.10:

$$\Phi(\mathbf{x})=\frac{1}{4\pi\varepsilon_0}\int_V \underbrace{\rho(\mathbf{x}')}_{=0} G(\mathbf{x},\mathbf{x}')d^3x'-\frac{1}{4\pi}\oint_S \Phi(\mathbf{x}')\frac{\partial}{\partial n'}G(\mathbf{x},\mathbf{x}')da' \quad (1.44)$$

There is no charge inside. $\Rightarrow \Phi(\mathbf{x})=-\dfrac{1}{4\pi}\oint_S \Phi(\mathbf{x}')\dfrac{\partial}{\partial n'}G(\mathbf{x},\mathbf{x}')da'$

Note: The unit vector $\mathbf{n}'$ is normal to the surface and pointing outward from volume of interest. $\frac{\partial}{\partial n'}$ is a differentiation along $\mathbf{n}'$ ($\frac{\partial}{\partial n'}=\frac{\partial}{\partial r'}$ for this example).

Rewrite (3.125):

$$G(\mathbf{x},\mathbf{x}') = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi)}{(2l+1)\left[1-(\frac{a}{b})^{2l+1}\right]} \left(r_<^l - \frac{a^{2l+1}}{r_<^{l+1}} \right)\left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}} \right)$$

For this example, $a=0$, $r_> = r'$, and $r_< = r$, hence

$$G(\mathbf{x},\mathbf{x}') = 4\pi \sum_{l,m} \frac{1}{2l+1} Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi) r^l \left(\frac{1}{r'^{l+1}} - \frac{r'^l}{b^{2l+1}}\right)$$

$$\Rightarrow \frac{\partial G}{\partial r'} = 4\pi \sum_{l,m} \frac{1}{2l+1} Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi) r^l \left(-\frac{l+1}{r'^{l+2}} - \frac{l r'^{l-1}}{b^{2l+1}}\right)$$

$$\Rightarrow \left.\frac{\partial G}{\partial n'}\right|_{r'=b} = \left.\frac{\partial G}{\partial r'}\right|_{r'=b} = -\frac{4\pi}{b^2} \sum_{l,m} (\tfrac{r}{b})^l Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi) \qquad (7)$$

$$da' = r'^2 \sin\theta' d\theta' d\varphi' = b^2 d\Omega' \qquad (8)$$

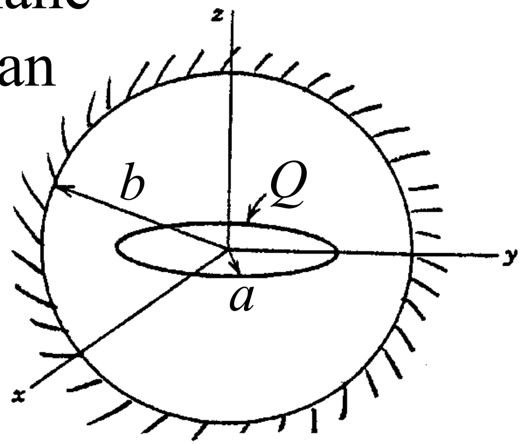
$$\left.\Phi(\mathbf{x}')\right|_S = \Phi(r'=b) = V(\theta',\varphi') \qquad (9)$$

Substituting (7) - (9) into $\Phi(\mathbf{x}) = -\frac{1}{4\pi}\oint_S \Phi(\mathbf{x}')\frac{\partial}{\partial n'}G(\mathbf{x},\mathbf{x}')da'$, we get

$$\Phi(\mathbf{x}) = \sum_{l,m} \left[\int V(\theta',\varphi')Y_{lm}^*(\theta',\varphi')d\Omega'\right] (\tfrac{r}{b})^l Y_{lm}(\theta,\varphi) \qquad (3.128)$$

Example 2: Potential due to a uniformly charged ring of radius a and total charge Q located on the x-y plane inside a grounded conducting sphere of radius b

Sol: In spherical coordinates, the x-y plane is at $\theta = \pi/2$. The charge density $\rho(\mathbf{x})$ can be written as



$$\rho(\mathbf{x}) = \frac{Q}{2\pi a^2}\delta(r-a)\delta(\cos\theta) \qquad (3.129)$$

The potental is given by

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\mathbf{x}')G(\mathbf{x},\mathbf{x}')d^3x' - \frac{1}{4\pi}\oint_S \underbrace{\Phi(\mathbf{x}')}_{=0}\frac{\partial}{\partial n'}G(\mathbf{x},\mathbf{x}')da' \qquad (1.44)$$

There is no inner conductor in this problem. ⇒ (3.125) reduces to

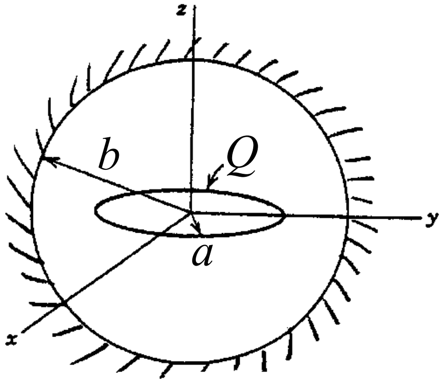
$$G(\mathbf{x},\mathbf{x}') = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi)r_<^l\left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right) \quad (10)$$

3.10 Solution of Potential Problems with the Spherical Green Function Expansion (*continued*)

Symmetry in $\varphi \Rightarrow m=0$. Hence,

$$Y_{lm}(\theta,\varphi) \to Y_{l0}(\theta,\varphi)=\sqrt{\frac{2l+1}{4\pi}}P_l(\cos\theta)$$

$$\Rightarrow \quad G(\mathbf{x},\mathbf{x}')=\sum_{l=0}^{\infty} P_l(\cos\theta')P_l(\cos\theta)r_<^l\left(\frac{1}{r_>^{l+1}}-\frac{r_>^l}{b^{2l+1}}\right) \qquad (11)$$



Substiting (11) and $\rho(\mathbf{x})=\dfrac{Q}{2\pi a^2}\delta(r-a)\delta(\cos\theta)$ into (1.44), we obtain

$$\Phi(\mathbf{x})=\frac{1}{4\pi\varepsilon_0}\int d^3x'\rho(\mathbf{x}')G(\mathbf{x},\mathbf{x}')$$

$$=\frac{Q}{8\pi^2\varepsilon_0 a^2}\int r'^2dr'd\cos\theta' d\varphi'\left[\begin{array}{l}\delta(r'-a)\delta(\cos\theta')\\ \cdot\sum\limits_{l=0}^{\infty} P_l(\cos\theta')P_l(\cos\theta)r_<^l\left(\dfrac{1}{r_>^{l+1}}-\dfrac{r_>^l}{b^{2l+1}}\right)\end{array}\right]$$

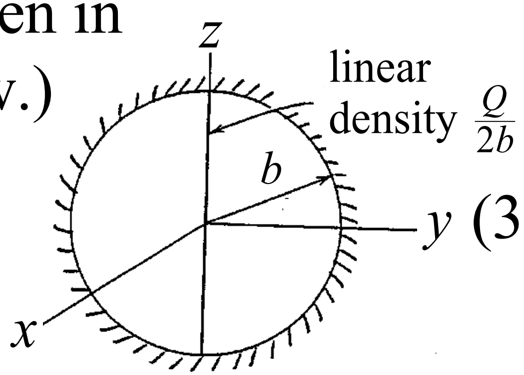
$$=\frac{Q}{4\pi\varepsilon_0}\sum_{l=0}^{\infty} P_l(0)P_l(\cos\theta)r_<^l\left(\frac{1}{r_>^{l+1}}-\frac{r_>^l}{b^{2l+1}}\right) \qquad (3.130)$$

where $r_<$ $(r_>)$ is the smaller (larger) of r and a.

3.10 Solution of Potential Problems with the Spherical Green Function Expansion (*continued*)

Example 3: Potential due to a uniformly charged line of length $2b$ and total charge Q located on the z-axis inside a grounded conducting sphere of radius b (see figure)

Sol: The charge density $\rho(\mathbf{x})$ can be written in spherical coordinates as (see problem below.)



$$\rho(\mathbf{x}) = \frac{Q}{2b}\frac{1}{2\pi r^2}\left[\delta(\cos\theta - 1) + \delta(\cos\theta + 1)\right] \qquad (3.132)$$

The potential is given by

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\mathbf{x}')G(\mathbf{x},\mathbf{x}')d^3x' - \frac{1}{4\pi}\oint_S \underbrace{\Phi(\mathbf{x}')}_{=0}\frac{\partial}{\partial n'}G(\mathbf{x},\mathbf{x}')da' \qquad (1.44)$$

Rewrite (11), which is applicable to this problem:

$$G(\mathbf{x},\mathbf{x}') = \sum_{l=0}^{\infty} P_l(\cos\theta')P_l(\cos\theta)r_<^l\left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right) \qquad (11)$$

Substituting (11) into (1.44), we obtain

3.10 Solution of Potential Problems with the Spherical Green Function Expansion (*continued*)

$$\Phi(\mathbf{x}) = \frac{Q}{8\pi\varepsilon_0 b}\int r'^2 dr' d\cos\theta' d\varphi' \left[\begin{array}{l} \dfrac{\delta(\cos\theta'-1)+\delta(\cos\theta'+1)}{2\pi r'^2} \\ \cdot \displaystyle\sum_{l=0}^{\infty} P_l(\cos\theta')P_l(\cos\theta)\, r_<^l \left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right) \end{array}\right]$$

$$= \frac{Q}{8\pi\varepsilon_0 b}\sum_{l=0}^{\infty}\left[P_l(1)+P_l(-1)\right]P_l(\cos\theta)\,\underbrace{\int_0^b r_<^l\left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right)dr'}_{\uparrow} \qquad (3.133)$$

$$\boxed{\begin{aligned} &= \left(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}}\right)\int_0^r r'^l dr' + r^l\int_r^b \left(\frac{1}{r'^{l+1}} - \frac{r'^l}{b^{2l+1}}\right)dr' \\ &= \frac{2l+1}{l(l+1)}\left[1-\left(\tfrac{r}{b}\right)^l\right] \end{aligned}}$$

$P_l(-1) = (-1)^l$ and $P_l(1) = 1 \;\Rightarrow\;$ Odd l terms cancel.

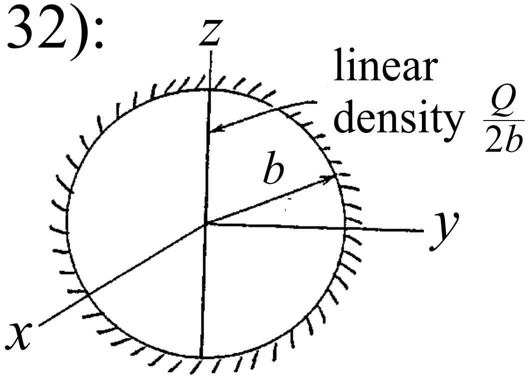
$$\Rightarrow \Phi(\mathbf{x}) = \frac{Q}{4\pi\varepsilon_0 b}\left\{\ln\left(\tfrac{b}{r}\right) + \sum_{j=1}^{\infty}\frac{4j+1}{2j(2j+1)}\left[1-\left(\tfrac{r}{b}\right)^{2j}\right]P_{2j}(\cos\theta)\right\} \quad (3.136)$$

Note that the $l = 0$ term in (3.133) is given by $\ln\left(\tfrac{b}{r}\right)$. See p.124.

Example 3.1: Show that the charge density in (3.132):

$$\rho(\mathbf{x}) = \frac{Q}{2b}\frac{1}{2\pi r^2}[\delta(\cos\theta-1)+\delta(\cos\theta+1)]$$

represents a unifom charge distribution along z.



Solution: The total charge is

$$\int\rho(\mathbf{x})d^3x = \frac{Q}{2b}\int_0^b r^2dr\int_{-1}^1 d\cos\theta\int_0^{2\pi}d\varphi\frac{\delta(\cos\theta-1)+\delta(\cos\theta+1)}{2\pi r^2}$$

$$= \frac{Q}{2b}\int_0^b dr\int_{-1}^1 d\cos\theta[\underbrace{\delta(\cos\theta-1)}_{\theta=0,\ +z\text{-axis}} + \underbrace{\delta(\cos\theta+1)}_{\theta=\pi,\ -z\text{-axis}}]$$

$$= \frac{Q}{2b}\int_{-b}^b dz \Rightarrow \text{uniform distribution from } z=-b \text{ to } z=b.$$

Note: The above integration over $\cos\theta$ starts from $\cos\theta=-1$ and and ends at $\cos\theta=1$. It does not cross 1 or -1. This issue can be resolved by a limiting procedure; namely, we write

$$\delta(\cos\theta-1)+\delta(\cos\theta+1) = \lim_{\varepsilon\to 0}[\delta(\cos\theta-1+\varepsilon)+\delta(\cos\theta+1-\varepsilon)]$$

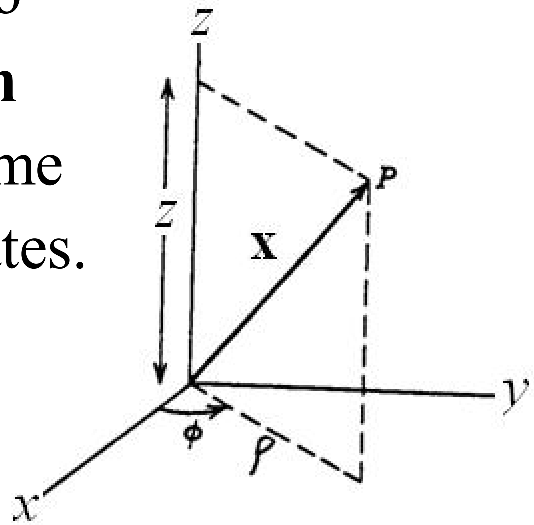
Self-learning

3.11 Expansion of Green Functions in Cylindrical Coordinates

Consider the Green equation:

$$\nabla^2 G(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}'),\ \text{with } G(\mathbf{x},\mathbf{x}') = 0 \text{ as } |\mathbf{x}| \to \infty$$

An obvious solution is $1/|\mathbf{x}-\mathbf{x}'|$. We have also solved this equation by the **method of expansion** in spherical coordinates [(3.70)]. Here, by the same method, we solve it again in cylindrical coordinates.



Write $\delta(\mathbf{x}-\mathbf{x}')$ as

$$\delta(\mathbf{x}-\mathbf{x}') = \tfrac{1}{\rho}\delta(\rho-\rho')\delta(\varphi-\varphi')\delta(z-z')$$

with
$$\begin{cases} \delta(\varphi-\varphi') = \frac{1}{2\pi}\sum\limits_{m=-\infty}^{\infty} e^{im(\varphi-\varphi')} \\ \delta(z-z') = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk e^{ik(z-z')} = \frac{1}{\pi}\int_0^{\infty} dk \cos[k(z-z')] \end{cases}$$

$$\Rightarrow \nabla^2 G(\mathbf{x},\mathbf{x}') = -\frac{2}{\pi}\frac{\delta(\rho-\rho')}{\rho}\sum_{m=-\infty}^{\infty}\int_0^{\infty} dk e^{im(\varphi-\varphi')}\cos[k(z-z')] \qquad (12)$$

Self-learning

3.11 Expansion of Green Functions in Cylindrical Coordinates (*continued*)

Since $e^{im\varphi}$ and e^{ikz} are complete sets, we may expand $G(\mathbf{x},\mathbf{x}')$ in variables φ and z

$$G(\mathbf{x},\mathbf{x}')=\frac{1}{2\pi}\frac{1}{\pi}\sum_{m=-\infty}^{\infty}\int_0^\infty dk g_m(k,\rho,\rho')e^{im(\varphi-\varphi')}\cos[k(z-z')] \qquad (3.140)$$

where the coefficient $g_m(k,\rho,\rho')$ is a function of m, k, ρ and ρ'.

Substituting (3.140) into (12) we get

$$\frac{1}{2\pi^2}\sum_{m=-\infty}^{\infty}\int_0^\infty dk\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}+\frac{\partial^2}{\partial z^2}\right)$$

$$\cdot g_m(k,\rho,\rho')e^{im(\varphi-\varphi')}\cos[k(z-z')]$$

$$=-\frac{2}{\pi}\frac{\delta(\rho-\rho')}{\rho}\sum_{m=-\infty}^{\infty}\int_0^\infty dk e^{im(\varphi-\varphi')}\cos[k(z-z')] \qquad (13)$$

In (13), $\frac{\partial^2}{\partial\varphi^2}\rightarrow -m^2$, $\frac{\partial^2}{\partial z^2}\rightarrow -k^2$, $\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}=\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}$. Hence,

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}-\left(k^2+\frac{m^2}{\rho^2}\right)\right]g_m(k,\rho,\rho')=-\frac{4\pi}{\rho}\delta(\rho-\rho') \qquad (3.141)$$

Self-learning

3.11 Expansion of Green Functions in Cylindrical Coordinates (*continued*)

Solutions for modified Bessel functions. See (3.98)-(3.101) in Jackson.

$$\Rightarrow g_m(k,\rho,\rho') = \begin{cases} AI_m(k\rho) + BK_m(k\rho), & \rho < \rho' \\ A'I_m(k\rho) + B'K_m(k\rho), & \rho > \rho' \end{cases}$$

(i) g_m is finite at $\rho = 0. \Rightarrow B = 0$

(ii) g_m remains finite as $\rho \to \infty. \Rightarrow A' = 0$

(iii) g_m is continuous at $\rho = \rho'$.

$$\Rightarrow \; AI_m(k\rho') = B'K_m(k\rho')$$

$$\Rightarrow \frac{A}{B'} = \frac{K_m(k\rho')}{I_m(k\rho')} \Rightarrow \begin{cases} A = CK_m(k\rho') \\ B' = CI_m(k\rho') \end{cases}$$

$$\Rightarrow g_m(k,\rho,\rho') = \begin{cases} CK_m(k\rho')I_m(k\rho), & \rho < \rho' \\ CI_m(k\rho')K_m(k\rho), & \rho > \rho' \end{cases}$$

$$= CI_m(k\rho_<)K_m(k\rho_>) \qquad (14)$$

Self-learning

(iv) To obtain the coefficient C in $g_m(k,\rho,\rho')=CI_m(k\rho_<)K_m(k\rho_>)$, mutiply (3.141) by ρ and integrate form $\rho'-\varepsilon$ to $\rho'+\varepsilon$ $(\varepsilon\to 0)$

$$\left.\frac{dg_m}{d\rho}\right|_{\rho'+\varepsilon}-\left.\frac{dg_m}{d\rho}\right|_{\rho'-\varepsilon}=-\frac{4\pi}{\rho'}$$

$$\Rightarrow Ck[I_m(k\rho')K_m'(k\rho')-K_m(k\rho')I_m'(k\rho')]=-\frac{4\pi}{\rho'}$$

Use the relation: $I_m(x)K_m'(x)-I_m'(x)K_m(x)=-1/x$ (3.147)

$$\Rightarrow Ck(\frac{-1}{k\rho'})=-\frac{4\pi}{\rho'}\Rightarrow C=4\pi\Rightarrow g_m(k,\rho,\rho')=4\pi I_m(k\rho_<)K_m(k\rho_>)$$

Substituting the above expression for $g_m(k,\rho,\rho')$ into (3.140)

$$\Rightarrow G(\mathbf{x},\mathbf{x}')=\frac{2}{\pi}\sum_{m=-\infty}^{\infty}\int_0^\infty dk e^{im(\varphi-\varphi')}\cos\left[k(z-z')\right]I_m(k\rho_<)K_m(k\rho_>)$$

Since $G(\mathbf{x},\mathbf{x}')=\frac{1}{|\mathbf{x}-\mathbf{x}'|}$, we have by the uniqueness theorem

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|}=\frac{2}{\pi}\sum_{m=-\infty}^{\infty}\int_0^\infty dk e^{im(\varphi-\varphi')}\cos\left[k(z-z')\right]I_m(k\rho_<)K_m(k\rho_>) \quad (3.148)$$

3.12 Eigenfunction Expansion for Green Functions

Eigenfunction Expansion of Green Function in 3 Dimensions :

We have obtained the Green function for the Poisson equation by the method of eigenfunction expansion in 2 dimensions [e.g., (3.118), in θ, φ]. Here, we develop a general technique to obtain the Green function by eigenfunction expansion in 3 dimensions. Consider the Green function for a more general equation (with homogeneous b.c.):

$$\nabla^2 G(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}') \tag{3.156s}$$

We shall solve (3.156s) by expanding $G(\mathbf{x},\mathbf{x}')$ and $\delta(\mathbf{x}-\mathbf{x}')$ in eigenfunctions of a related problem formulated as follows.

an eigenvalue to be determined by the b.c.

$$\nabla^2 \psi(\mathbf{x}) + \lambda\psi(\mathbf{x}) = 0 \tag{3.153s}$$

with the same boundary surface and homogeneous b.c. as for (3.156s).

3.12 Eigenfunction Expansion for Green Functions (*continued*)

Assume the eigenfunctions and eigenvalues are ψ_n and λ_n, respectively.

$$\nabla^2\psi_n(\mathbf{x})+\lambda_n\psi_n(\mathbf{x})=0.$$

Since the operator $[\nabla^2]$ is Hermitian, we have

The element in the *i*-th row and *j*-th column is equal to the complex conjugate of the element in the *j*-th row and *i*-th column.

$$\int_V \psi_m^*(\mathbf{x})\psi_n(\mathbf{x})d^3x=\delta_{mn}$$

and ψ_n form a complete set with *real* eigenvalue λ_n [see Apendix A].

Write $G(\mathbf{x},\mathbf{x}')=\sum\limits_n a_n(\mathbf{x}')\psi_n(\mathbf{x})$ (3.157)

Substituting (3.157) and $\delta(\mathbf{x}-\mathbf{x}')=\sum\limits_n \psi_n^*(\mathbf{x}')\psi_n(\mathbf{x})$ [see (2.35)] into

$\nabla^2 G(\mathbf{x},\mathbf{x}')=-4\pi\delta(\mathbf{x}-\mathbf{x}')$, we obtain

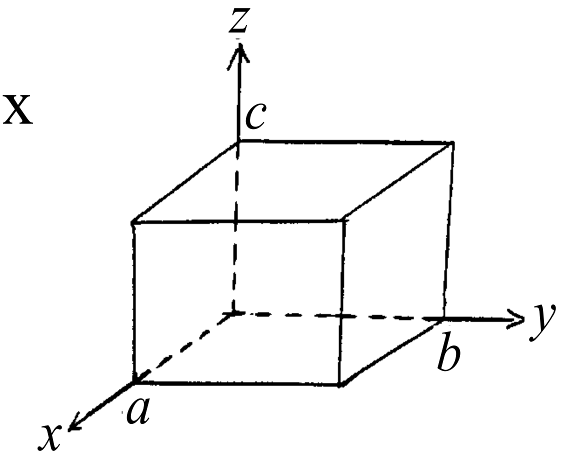
$$\sum_n a_n(\mathbf{x}')\nabla^2\psi_n(\mathbf{x})=-4\pi\sum_n \psi_n^*(\mathbf{x}')\psi_n(\mathbf{x})$$

Since $\nabla^2\psi_n(\mathbf{x})=-\lambda_n\psi_n(\mathbf{x})$, we have

$$\sum_n a_n(\mathbf{x}')(-\lambda_n)\psi_n(\mathbf{x})=-4\pi\sum_n \psi_n^*(\mathbf{x}')\psi_n(\mathbf{x})$$

$$\Rightarrow\ a_n(\mathbf{x}')=4\pi\frac{\psi_n^*(\mathbf{x}')}{\lambda_n}\Rightarrow G(\mathbf{x},\mathbf{x}')=4\pi\sum_n\frac{\psi_n^*(\mathbf{x}')\psi_n(\mathbf{x})}{\lambda_n} \qquad (3.160\text{s})$$

We now specialize to the Green function for the Poisson equation i.e., (3.156s).



Example 1: Green function for a rectangular box

$$\nabla^2 G(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}')$$

$$\text{with } G(\mathbf{x},\mathbf{x}') = 0 \text{ at } \begin{cases} x = 0 \text{ and } a \\ y = 0 \text{ and } b \\ z = 0 \text{ and } c \end{cases}$$

Sol: Consider the corresponding eigenvalue problem [(3.153s) with $\lambda \to k^2$]: $\nabla^2\psi(\mathbf{x}) + k^2\psi(\mathbf{x}) = 0$ with the same b.c.

$$\text{Let } \psi(\mathbf{x}) = X(x)Y(y)Z(z) \Rightarrow \underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{-k_l^2} + \underbrace{\frac{1}{Y}\frac{d^2Y}{dy^2}}_{-k_m^2} + \underbrace{\frac{1}{Z}\frac{d^2Z}{dz^2}}_{-k_n^2} + k^2 = 0$$

$$\Rightarrow \begin{cases} X(x) = Ae^{ik_l x} + Be^{-ik_l x} \\ Y(x) = Be^{ik_m y} + Ce^{-ik_m y} \\ Z(x) = De^{ik_n z} + Ee^{-ik_n z} \end{cases} \text{ with } k^2 = k_l^2 + k_m^2 + k_n^2$$

3.12 Eigenfunction Expansion for Green Functions (*continued*)

$$\text{b.c.}\ \begin{cases} X(x)=0 \text{ at } x=0 \text{ and } a \\ Y(x)=0 \text{ at } y=0 \text{ and } b \\ Z(x)=0 \text{ at } z=0 \text{ and } c \end{cases} \Rightarrow \begin{cases} k_l=\frac{l\pi}{a} \\ k_m=\frac{m\pi}{b} \\ k_n=\frac{n\pi}{c} \end{cases} \text{ and } \begin{cases} X=\sin\frac{l\pi x}{a}, \\ Y=\sin\frac{m\pi y}{b}, \\ Z=\sin\frac{n\pi z}{c}, \end{cases}$$

$$\Rightarrow k^2=k_{lmn}^2=\pi^2\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}\right)$$

$$\Rightarrow \psi(\mathbf{x})=\sqrt{\frac{8}{abc}}\sin\frac{l\pi x}{a}\sin\frac{m\pi y}{b}\sin\frac{n\pi z}{c}$$

Substituting $\psi(\mathbf{x})$ into (3.160): $G(\mathbf{x},\mathbf{x}')=4\pi\sum_j\frac{\psi_j^*(\mathbf{x}')\psi_j(\mathbf{x})}{\lambda_j}$, we obtain

$$\boxed{\sum_j \to \sum_{l,m,n}\ ;\ \lambda_j \to k_{lmn}^2}$$

$$G(\mathbf{x},\mathbf{x}')$$

$$=\frac{32}{\pi abc}\sum_{l,m,n=1}^{\infty}\frac{\sin\frac{l\pi x'}{a}\sin\frac{l\pi x}{a}\sin\frac{m\pi y'}{b}\sin\frac{m\pi y}{b}\sin\frac{n\pi z'}{c}\sin\frac{n\pi z}{c}}{\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}} \quad (3.167)$$

Example 2: Green function for infinite space

$$\nabla^2 G(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}') \text{ with } G(\mathbf{x},\mathbf{x}') = 0 \text{ as } |\mathbf{x}| \to \infty$$

Sol: Consider the corresponding eigenvalue problem

$$\nabla^2\psi(\mathbf{x}) + k^2\psi(\mathbf{x}) = 0 \quad \text{in infinite space}$$

The solution is

$$\psi_{\mathbf{k}}(\mathbf{x}) = \frac{1}{(2\pi)^{3/2}} e^{i\mathbf{k}\cdot\mathbf{x}} = \frac{1}{(2\pi)^{3/2}} e^{ik_x x + ik_y y + ik_z z} \qquad (3.162)$$

where $\mathbf{k} = k_x\mathbf{e}_x + k_y\mathbf{e}_y + k_z\mathbf{e}_z$

Since the region of interest is infinite space, we have continuous engenvalue k^2 and the factor $1/(2\pi)^{3/2}$ gives the normalization condition for $\psi_{\mathbf{k}}(\mathbf{x})$:

$$\int \psi^*_{\mathbf{k}'}(\mathbf{x})\psi_{\mathbf{k}}(\mathbf{x}) d^3x = \delta(\mathbf{k}-\mathbf{k}') \qquad (3.163)$$

[see p.69 for a one dimensional example.]

Self-learning

3.12 Eigenfunction Expansion for Green Functions (*continued*)

So the series expansion: $G(\mathbf{x},\mathbf{x}') = 4\pi\sum_n \frac{\psi_n^*(\mathbf{x}')\psi_n(\mathbf{x})}{\lambda_n-\lambda}$ [(3.160)] becomes

$$G(\mathbf{x},\mathbf{x}') = 4\pi\int \frac{\psi_{\mathbf{k}}^*(\mathbf{x}')\psi_{\mathbf{k}}(\mathbf{x})}{\lambda_{\mathbf{k}}-\lambda} d^3k$$

With $\lambda_{\mathbf{k}} = k^2$, $\lambda = 0$, and $\psi_{\mathbf{k}} = \frac{1}{(2\pi)^{3/2}} e^{i\mathbf{k}\cdot\mathbf{x}}$, we have

$$G(\mathbf{x},\mathbf{x}') = \frac{1}{2\pi^2}\int d^3k \frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}}{k^2}$$

Since $G(\mathbf{x},\mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|}$, we get another mathematical expression for $\frac{1}{|\mathbf{x}-\mathbf{x}'|}$ by the uniquess theorem

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{1}{2\pi^2}\int d^3k \frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}}{k^2} \qquad (3.164)$$

Solution of Inhomogeneous Differential Equation by the Green Function Method :

To show the usefulness of the 3-dimensional Green function just obtained, we consider an inhomogeneous differential equation:

$$\nabla^2 u(\mathbf{x}) + [f(\mathbf{x}) + \lambda] u(\mathbf{x}) = -4\pi S(\mathbf{x}) \qquad (15)$$

wth homogeneous b.c. In (15), $S(\mathbf{x})$ is a distributed source. We have shown that the solution for the same equation with a point source:

$$\nabla^2 G(\mathbf{x}, \mathbf{x}') + [f(\mathbf{x}) + \lambda] G(\mathbf{x}, \mathbf{x}') = -4\pi\delta(\mathbf{x} - \mathbf{x}') \qquad (3.156)$$

is
$$G(\mathbf{x}, \mathbf{x}') = 4\pi \sum_n \psi_n^*(\mathbf{x}')\psi_n(\mathbf{x})/(\lambda_n - \lambda), \qquad (3.160)$$

where $\psi_n(\mathbf{x})$ is the eigenfunction of $\nabla^2\psi_n(\mathbf{x}) + [f(\mathbf{x}) + \lambda_n]\psi_n(\mathbf{x}) = 0.$

Then, the solution of (15) is $u(\mathbf{x}) = \int_V G(\mathbf{x}, \mathbf{x}')S(\mathbf{x}')d^3x',$ (16)

which can be verified if we operate both sides with $\nabla^2 + f(\mathbf{x}) + \lambda$ and apply (3.156) to the RHS.

Note: If $\lambda = \lambda_n$, there is no solution unless $\int_V u_n^*(\mathbf{x})S(\mathbf{x})d^3x = 0.$

Homework of Chap. 3

Problem 3.1

Two concentric spheres have radii a, b $(b > a)$ and each is divided into two hemispheres by the same horizontal plane. The upper hemisphere of the inner sphere and the lower hemisphere of the outer sphere are maintained at potential V. The other hemispheres are at zero potential.

Determine the potential in the region $a \le r \le b$ as a series in Legendre polynomials. Include terms at least up to $l = 4$. Check your solution against known results in the limiting cases $b \to \infty$, and $a \to 0$.

Problem 3.2

A spherical surface of radius R has charge uniformly distributed over its surface with a density $Q/4\pi R^2$, except for a spherical cap at the north pole, defined by the cone $\theta = \alpha$.

(a)Show that the potential inside the spherical surface can be expressed as

$$\Phi = \frac{Q}{8\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{1}{2l+1}\left[P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)\right]\frac{r^l}{R^{l+1}}P_l(\cos\theta)$$

where, for $l = 0$, $P_{l-1}(\cos\,\alpha) = -1$. What is the potential outside?

(b) Find the magnitude and the direction of the electric field at the origin.

(c) Discuss the limiting forms of the potential (part a) and electric field (part b) as the spherical cap becomes (1) very small, and (2) so large that the area with charge on it becomes a very small cap at the south pole.

Problem 3.3

A thin, flat, conducting, circular disc of radius R is located in the $x-y$ plane with its center at the origin, and is maintained at a fixed potential V. With the information that the charge density on a disc at fixed potential is proportional to $(R^2 - \rho^2)^{-1/2}$, where ρ is the distance out from the center of the disc,

(a)show that for $r > R$ the potential is

$$\Phi(r,\theta,\phi) = \frac{2V}{\pi}\frac{R}{r}\sum_{l=0}^{\infty}\frac{(-1)^l}{2l+1}\left(\frac{R}{r}\right)^{2l} P_{2l}(\cos\theta)$$

(b) fond the potential for $r < R$.

(c) What is the capacitance of the disc?

Homework of Chap. 3

Problem 3.6

Two point charges q and $-q$ are located on the z axis at $z = +a$ and $z = -a$, respectively.

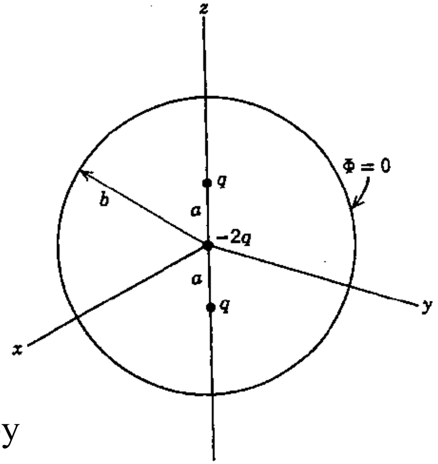
(a) Find the electrostatic potential as an expansion in sphereical harmonics and powers of r for both $r > a$ and $r < a$.

(b) Keeping the product $qa = p/2$ constant, take the limit of $a \to 0$ and find the potential for $r \neq 0$. This is by definition a dipole along the z axis and its potential.

(c) Suppose now that the dipole of part b is surrounded by a *grounded* spherical shell of radius b concentric with the origin. By linear superposition find the potential everywhere inside the shell.

Problem 3.7

Three point charges $(q, -2q, q)$ are located in a straight line with separation a and with the middle charge $(-2q)$ at the origin of a grounded conducting spherical shell of radius b, as indicated in the sketch.



(a) Write down the potential of the three charges in the absence of the grounded sphere. Find the limiting form of the potential as $a \to 0$, but the product $qa^2 = Q$ remains finite. Write this latter answer in spherical coordinates.

(b) The presence of the grounded sphere of radius b alters the potential for $r < b$. The added potential can be viewed as caused by the surface-charge density induced on the inner surface at $r = b$ or by image charges located at $r > b$. Use linear superposition to satisfy the boundary conditions and find the potential everywhere inside the sphere for $r < a$ and $r > a$. Show that in the limit $a \to 0$,

$$\Phi(r, \theta, \varphi) \to \frac{Q}{2\pi\varepsilon_0 r^3}\left(1 - \frac{r^5}{b^5}\right) P_2(\cos\theta)$$

Problem 3.9

A hollow right circular cylinder of radius b has its axis coincident with the z axis and its ends at $z = 0$ and $z = L$. The potential on the end *faces* is zero, while the potential on the cylindrical surface is given as $V(\phi, z)$. Using the appropriate separation of variables in cylindrical coordinates, find a series solution for the potential anywhere inside the cylinder.

Homework of Chap. 3

Problem 3.17

The Dirichlet Green function for the unbounded space between the planes at $z = 0$ and $z = L$ allows discussion of a point charge or a distribution of charge between parallel conducting planes held at zero potential.

(a) Using cylindrical coordinates show that one form of the Green function is

$$G(x,x') = \frac{4}{L}\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} e^{im(\varphi-\varphi')}\sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z'}{L}\right)I_m\left(\frac{n\pi}{L}\rho_<\right)K_m\left(\frac{n\pi}{L}\rho_>\right)$$

(b)Show that an alternative form of the Green function is

$$G(x,x') = 2\sum_{m=-\infty}^{\infty}\int_0^{\infty} dk e^{im(\varphi-\varphi')}J_m(k\rho)J_m(k\rho')\frac{\sin(kz_<)\sinh\left[k(L-z_>)\right]}{\sinh(kL)}$$

Problem 3.20

(a)From the results of Problem 3.17 or from first principles show that the potential at a point charge q between two infinite parallel conducting planes geld at zero potential can be written as

$$\Phi(z,\rho) = \frac{q}{\pi\varepsilon_0 L}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi z_0}{L}\right)\sin\left(\frac{n\pi z}{L}\right)K_0\left(\frac{n\pi\rho}{L}\right)$$

where the planes are at $z = 0$ and $z = L$ and the charge is one the z axis at the point $z = z_0$.

(b)Calculate the induced surface-charge densities $\sigma_0(\rho)$ and $\sigma_L(\rho)$ on the lower and upper plates. The result for $\sigma_L(\rho)$ is

$$\sigma_L(\rho) = \frac{q}{L^2}\sum_{n=1}^{\infty}(-1)^n n\sin\left(\frac{n\pi z_0}{L}\right)K_0\left(\frac{n\pi\rho}{L}\right)$$

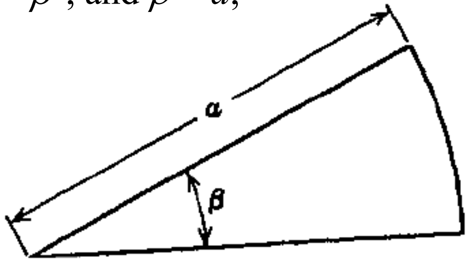
Discuss the connection of this expression with that of Problem 3.19b and 3.19c.

(c) From the answer in part b, calculate the total charge Q_L on the plate at $z = L$. By summing the Fourier series or by other means of comparison, check your answer against the known expression of Problem 1.13.

Homework of Chap. 3

Problem 3.22

The geometry of a two-dimensional potential problem is defined in polar coordinates by the surfaces $\phi = 0$, $\phi = \beta$, and $\rho = a$, as indicated in the sketch.



Using separation of variables in polar coordinates, show that the Green function can be written as

$$G\left(\rho,\varphi,\rho',\varphi'\right)=\sum_{m=1}^{\infty}\frac{4}{m}\rho_<^{\frac{m\pi}{\beta}}\left(\frac{1}{\rho_>^{\frac{m\pi}{\beta}}}-\frac{\rho_>^{\frac{m\pi}{\beta}}}{\rho^{2\frac{m\pi}{\beta}}}\right)\sin\left(\frac{m\pi\varphi}{\beta}\right)\sin\left(\frac{m\pi\varphi'}{\beta}\right)$$

Problem 2.25 may be of use.

Appendix A. Eigenvalue Problem

(Ref. Mathews and Walker "Math. Meth. in Phys." 2nd Ed., Ch. 9)

A Note on Terminology and Definitions :

(i) Linear differential operator: L is a linear differential operator if

$$L(au_1+bu_2)=aLu_1+bLu_2$$

(u_1 and u_2: arbitrary functions; a and b : arbitrary constants.)

Examples: $\frac{d^n}{dx^n}$, $\frac{d}{dx}p(x)\frac{d}{dx}-q(x)$

(ii) Linear D.E.: A D.E. is linear if it can be put in the form

$$\sum_{n=0}^{N} f_n(x)\frac{d^n u}{dx^n}=g(x)$$

i.e. The dependent variable u in all terms is of the first degree or zero degree.

(iii) Homogeneous linear D.E.: The above equation with $g(x)=0$, i.e. all terms are of the first degree in dependent variable u.

$\Rightarrow$ If u satisfies the D. E., so does au.

(iv) Homogeneous b.c.: If u satisfies the b.c., so does au.

(v) Homogeneous linear boundary-value problem: a problem governed by a homog. linear D.E with homog. b.c.'s.

$\Rightarrow$ If u satisfies the D.E. and b.c's, so does au.

$\Rightarrow$ Any linear combination of u_n ($f = \sum_n a_n u_n$) is also a solution.

Note: A problem can be inhomogeneous because either the b.c. or the D.E. is inhomogeneous (see M&W, p. 218.)

Formulation of an Eigenvalue Problem:

An eigenvalue problem involving the differential operator* consists of

$$\underbrace{\text{a linear homog. D.E. of the form}} + \underbrace{\text{homog. b.c.'s of the form}}$$

$$Lu(x) = \lambda\rho(x)u(x), \;\; a \le x \le b$$

L : linear differential operator

λ : eigenvlue

u : eigenfunction

$\rho(x)$: density function, a given function of real value

$u(a) = 0$ & $u(b) = 0$

or $u'(a) = 0$ & $u'(b) = 0$

or $u(a) = u(b)$ & $u'(a) = u'(b)$

or $u(a)$ & $u(b)$ are finite

*There are also eigenvalue problems which involve the matrix or integral operator.

Definition of Hermitian Operator :

An operator L is a <u>Hermitian</u> if

$$\int_a^b u_1^*(x) L u_2(x)dx = \left[\int_a^b u_2^*(x) L u_1(x) dx\right]^*, \qquad \text{(A.1)}$$

where u_1 and u_2 are arbitrary functions obeying the homog. b.c.'s.

Example 1: $L = \dfrac{d^2}{dx^2}$

integration by parts

$$\int_a^b u_1^*(x)\frac{d^2}{dx^2}u_2(x)dx = \cancel{u_1^*\frac{d}{dx}u_2\Big|_a^b} - \int_a^b \frac{du_1^*}{dx}\frac{du_2}{dx}dx$$

$$= \cancel{-u_2\frac{du_1^*}{dx}\Big|_a^b} + \int_a^b u_2\frac{d^2}{dx^2}u_1^* dx$$

$$= \left[\int_a^b u_2^*\frac{d^2}{dx^2}u_1 dx\right]^*$$

Appendix A. Eigenvalue Problem (*continued*)

Example 2: $L = \frac{d}{dx} p(x) \frac{d}{dx} - q(x)$ [Sturm-Liouville differential operator]

real function of x

$$\int_a^b u_1^* L u_2 dx = \int_a^b u_1^* \frac{d}{dx}\left(p \frac{d}{dx} u_2\right) dx - \int_a^b q u_1^* u_2 dx$$

$$= \cancel{u_1^* p \frac{d}{dx} u_2 \Big|_a^b} - \int_a^b \frac{du_2}{dx} p \frac{du_1^*}{dx} dx - \int_a^b q u_1^* u_2 dx$$

$$= \cancel{-u_2 p \frac{d}{dx} u_1^* \Big|_a^b} + \int_a^b u_2 \frac{d}{dx}\left(p \frac{d}{dx} u_1^*\right) dx - \int_a^b q u_1^* u_2 dx$$

$$= \left[\int_a^b u_2^* \frac{d}{dx}\left(p \frac{d}{dx} u_1\right) dx - \int_a^b q u_2^* u_1 d\right]^*$$

Properties of Eienvalue Problem with Hermitian Operator :

1. L is Hermitian $\Rightarrow \lambda_n$'s are real and u_n's are orthogonal

proof : Let u_i, u_j be eigenfunctions belonging to eigenvalues

λ_i, λ_j, respectively, i.e. $\begin{cases} Lu_i = \lambda_i \rho u_i \\ Lu_j = \lambda_j \rho u_j \end{cases}$

$$Lu_i = \lambda_i \rho u_i \Rightarrow \int_a^b u_j^* L u_i dx = \lambda_i \int_a^b u_j^* u_i \rho dx$$

Using the Hermitian property of L

$$\text{LHS} = \left[\int_a^b u_i^* L u_j dx\right]^* = \left[\lambda_j \int_a^b u_i^* u_j \rho dx\right]^* = \lambda_j^* \int_a^b u_i u_j^* \rho dx$$

$$\Rightarrow \left(\lambda_i - \lambda_j^*\right)\int_a^b u_i u_j^* \rho dx = 0$$

$$\Rightarrow \begin{cases} i = j \Rightarrow \lambda_i - \lambda_i^* = 0 \ \& \ \lambda_j - \lambda_j^* = 0 \Rightarrow \lambda_i \ \& \ \lambda_j \text{ are real.} \\ i \neq j \Rightarrow \int_a^b u_i u_j^* \rho dx = 0 \Rightarrow u_i \ \& \ u_j \text{ are orthogonal.} \end{cases} \qquad \text{(A.2)}$$

2. The eigenvalue problem is a linear and homogeneous boundary-value problem. So, if u_n's are solutions, $\sum\limits_n a_n u_n$ is also a solution.
3. If the differential operator of the eigenvalue problem is Hermitian, The eigenfunctions u_n form a complete set (see quotations below). Hence, any function can be expaned in terms of u_n. This makes the eigenfunctions powerful building blocks of physical quantities. Chapter 3 demonstrates how some electrostatic problems can be solved by the method of eigenfunction expansions.

"The eigenfunctions of a Hermitian operator form a complete set under very general conditions. We shall not prove this property here but it is in fact true for all the commonly encountered differential equations in physics." (M&W, p.265).

"All orthonormal sets of functions normally occurring in mathematical physics have been proved to be complete." (Jackson, p.68)

Examples of Eigenvalue Problem :

Example 1: $\frac{d^2X}{dx^2}=-\alpha^2 X$, b.c. $X(0)=0$ and $X(a)=0$ (Jackson, p. 70)

$$\Rightarrow X(x)=Ae^{i\alpha x}+Be^{-i\alpha x}$$

$$\begin{cases} X(0)=0\Rightarrow B=-A\Rightarrow X(x)=A(e^{i\alpha x}-e^{-i\alpha x})=A'\sin\alpha x \\ X(a)=0\Rightarrow \alpha=\alpha_n=\frac{\pi n}{a},\quad n=1,2,\ldots \qquad \text{eigenvalues} \end{cases}$$

$$\Rightarrow X(x)=\sum_n A_n \sin\alpha_n x \qquad \text{sum of eigenfunctions}$$

This simple example illustrates the following general properties of an eigenvalue problem:

a. Eigenvalues ($\alpha_n=n\pi/a$) are determined by the b.c.

b. The operator (d^2/dx^2) is Hermitian $\Rightarrow$ Eigenvalues are real.

c. The operator (d^2/dx^2) is Hermitian $\Rightarrow$ Eigenfunctions ($\sin\alpha_n x$) are orthogonal to each other and they form a complete set.

d. The D.E. & b.c. are both homogeneous. $\Rightarrow$ Eigenfunctions are only determined up to an arbitrary multiplicative factor A_n.

Appendix A. Eigenvalue Problem (*continued*)

Example 2: Eigenvalue problem involving the Legendre equation (Jackson Sec. 3.2 and 3.4, M&W Sec. 7.1)

$$\frac{d}{dx}\left[\left(1-x^2\right)\frac{du}{dx}\right]+\nu\left(\nu+1\right)u=0\ ,\ \ -1\leq x\leq 1,\ \text{b.c.}\ \begin{cases} u(-1)=\text{finite} \\ u(1)=\text{finite} \end{cases}$$

This is an eigenvalue problem of the form:

$$\overbrace{\left[\frac{d}{dx}\underset{\substack{\uparrow \\ \boxed{1-x^2}}}{p(x)}\frac{d}{dx}-\underset{\substack{\uparrow \\ \boxed{0}}}{q(x)}\right]}^{\text{Hermitian operator}}u(x)=\underset{\substack{\uparrow \\ \boxed{-\nu(\nu+1),\ \text{eigenvalue}}}}{\lambda}\overbrace{\rho(x)}^{1}u(x)$$

It is clear ν(or l) is the eigenvalue*. As shown in Sec. 3.2, the solution is

$$u(x)=P_l(x)\ \text{with}\ \nu=l=0,\ 1,\ 2,\ldots$$

*Strictly speaking, $-\nu(\nu+1)$ is the eigenvalue. But we shall loosely call ν an eigenvalue. (see M&W, p.262).

Since the operator is Hermitian, the eigenfunctions are orthogonal [see (A.2)]: $\int_a^b u_i(x)u_j^*(x)\rho(x)dx = 0, \quad \text{if } i \neq j$

Thus, $P_l(x)$ is orthogonal in index l and, with the density function $\rho(x) = 1,$ we have

$$\int_{-1}^{1} P_{l'}(x)P_l(x)dx = \tfrac{2}{2l+1}\delta_{l'l}, \qquad (3.21)$$

where the factor $\frac{2}{2l+1}$ is due to the fact that $P_l(x)$ is not normalized.

Also, the eigenfunctions of a Hermitian operator form a complete set. Hence, $P_l(x)$ is complete in index ℓ in the sense any function $f(x)$ can be expanded as

$$f(x) = \sum_{l=0}^{\infty} A_l P_l(x) \qquad (3.23)$$

Example 3: Eigenvalue problem involving the associated Legendre equation (Jackson Sec. 3.5, M&W Sec. 7.1)

$$\frac{d}{dx}(1-x^2)\frac{du}{dx}+[\nu(\nu+1)-\frac{m^2}{1-x^2}]u=0\ ,\quad -1\le x\le 1,\quad \begin{cases} u(-1)=\text{finite} \\ u(1)=\text{finite} \end{cases}$$

A question may arise as to whether ν or m is the eigenvalue. This can be resolved in the context of a physics problem: the Laplace equation in Sec. 3.1. There we have obtained the associated Legendre equation above, with m being the eigenvalue for a different equation:

$$\frac{d^2}{d\varphi^2}Q(\varphi)+m^2Q(\varphi)=0 \quad [(3.4)]$$

and, for $Q(\varphi)$ to be singled valued, m must be an integer.

Thus, for the associated Legendre equation we have here, ν is the eigenvalue while m is a fixed integer. As shown in Sec. 3.5 (p. 107), for u to be finite at $x=\pm 1$, the solution is

$$u(x)=P_l^m(x) \text{ with } \nu=l=|m|,\ |m|+1,\ |m|+2,\ldots$$

Appendix A. Eigenvalue Problem (*continued*)

Put the equation in the eigenvalue problem format, we have

$$\overbrace{[\frac{d}{dx}\underset{\substack{\uparrow\\ \boxed{1-x^2}}}{p(x)}\frac{d}{dx}-\underset{\substack{\uparrow\\ \boxed{m^2/(1-x^2)}}}{q(x)}]}^{\text{Hermitian operator}}u(x)=\underset{\substack{\nwarrow\\ \boxed{-\nu(\nu+1),\ \text{eigenvalue}}}}{\lambda}\,\overbrace{\rho(x)}^{1}u(x)$$

Since the operator is Hermitian, $P_l^m(x)$ is orthogonal in index ℓ and, with the density function $\rho(x)=1$, we have

$$\int_{-1}^{1}P_{l'}^m(x)P_l^m(x)dx=\frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}\delta_{ll'} \qquad (3.52)$$

Also, because of the Hermitian property of the operator, $P_l^m(x)$ is complete in index ℓ in the sense any function $f(x)$ can be expanded as

$$f(x)=\sum_{l=|m|}^{\infty}C_lP_l^m(x)\ \left[\text{see also M\&W, p.175.}\right] \qquad (A.3)$$

Appendix A. Eigenvalue Problem (*continued*)

Example 4: Eigenvalue problem involving the Bessel equation (Jackson Secs. 3.7 and 3.8; M&W Sec. 7.2)

$$\frac{d^2u}{dx^2}+\frac{1}{x}\frac{du}{dx}+\left(k^2-\frac{\nu^2}{x^2}\right)u=0\ ,\ \ 0\le x\le a\ ,\ \ \text{b.c.}\ \begin{cases} u(0)=\text{finite} \\ u(a)=0 \end{cases}$$

This equation can be written: $\dfrac{d}{dx}x\dfrac{du}{dx}+\left(k^2x-\dfrac{\nu^2}{x}\right)u=0$ (A.4)

In the context of the physics problem in Sec. 3.7, ν is fixed by (3.74). Hence, k is the eigenvalue and we have the solution:

$$u(x)=J_\nu(k_{\nu n}x),\ n=0,\ 1,\ 2,\cdots\ \ [k_{\nu n}:\ n\text{-th root of } J_\nu(k_{\nu n}a)=0]$$

Write (A.4) as $\Big[\overbrace{\frac{d}{dx}\underset{\uparrow\atop\boxed{x}}{p(x)}\frac{d}{dx}-\underset{\uparrow\atop\boxed{\nu^2/x}}{q(x)}}^{\text{Hermitian operator}}\Big]u(x)=\underset{\uparrow\atop\boxed{-k^2,\ \text{eigenvalue}}}{\lambda}\overbrace{\rho(x)}^{x\ \text{(density function)}}u(x)=0,$ we get

Orthogonalily: $\int_0^a J_\nu(k_{\nu n'}x)J_\nu(k_{\nu n}x)\underset{\nwarrow\ \boxed{\text{density function, see (A.2)}}}{x}dx=\frac{a^2}{2}[J_{\nu+1}(k_{\nu n}a)]^2\delta_{n'n}$ (3.95)

Completeness: $f(x)=\sum\limits_{n=1}^{\infty}C_nJ_\nu(k_{\nu n}x)$